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A generalization of Artinian modules

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Abstract. As a generalization of the Artinian module, we introduce the class of pseudo Artinian modules. We explore some algebraic properties of this class, and we study some topological properties of the prime spectrum of pseudo Artinian modules.

Keywords: Artinian modules, pseudo Artinian modules, prime submodule, Noetherian space, Spectral space.

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1. Introduction

Throughout the article, all rings are commutative with a nonzero identity and all modules are unitary. We recall some definitions.

Definition 1.1. Let M be an R-module and N be a submodule of M.

(1) $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and the *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(0_M :_R M)$. If there is no ambiguity, we will write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$).

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(2) N is said to be *prime* if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$), then $r \in (N : M)$ or $m \in N$. If N is prime, then the ideal $\mathfrak{p} := (N : M)$ is a prime ideal of R. In this case, N is said to be $\mathfrak{p}$-*prime* (see [13, 21]).

(3) The set of all prime submodules of M is called the *prime spectrum* of M and is denoted by $\mathrm{Spec}(M)$. Similarly, the collection of all $\mathfrak{p}$-prime submodules of M for any $\mathfrak{p} \in \mathrm{Spec}(R)$ is designated by $\mathrm{Spec}_{\mathfrak{p}}(M)$. Recall that an R-module M is said to be *primeless* if $\mathrm{Spec}(M) = \emptyset$.

(4) The set of all prime submodules of M containing N is denoted by $V^*(N)$ (see [22]). Following [16], we define $V(N)$ as

$$\{P \in \mathrm{Spec}(M) \mid (P : M) \supseteq (N : M)\}.$$

By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) of M. Set $\mathsf{Z}(M) = \{V(N) | N \leq M\}$. Then the elements of the set $\mathsf{Z}(M)$ satisfy the axioms for closed sets in a topological space $\mathrm{Spec}(M)$. The resulting topology due to $\mathsf{Z}(M)$ is called the *Zariski topology relative to* M and denoted by τ (see [16]).

In recent decades, the theory of prime submodules has been widely considered as a generalization of the theory of prime ideals in commutative rings. There are many articles that seek to generalize the various properties of the prime ideals of a ring to the prime submodules of a module (see [3, 5, 7, 8, 9, 10, 13]). To see a common generalization of the notion of prime submodule, we refer the reader to [11]. Also, there are some interesting applications of prime submodule theory in [10], where the authors show that an R-module M is Von-Neumann regular if and only if every submodule of M is an intersection of prime submodules of M.

The prime submodules of different types of modules were investigated by many researchers in the last decades. It is shown by Azizi in [4, Corollary 2.4] that any submodule N of an Artinian R-module M is prime if and only if $(N : M)$ is a maximal ideal of R. This is our motivation for the following definition.

Definition 1.2. An R-module M is said to be *pseudo Artinian* if either $\mathrm{Spec}(M) = \emptyset$ or $\mathrm{Spec}(M) \neq \emptyset$ and for each prime submodule P of M, $(P : M)$ is a maximal ideal of R.

In the next section, we show that the class of pseudo Artinian modules is more extensive than the class of Artinian modules. In Lemma 2.2, we will present some properties of pseudo Artinian modules. Theorem 2.7 explicitly expresses the radical of specific submodules of pseudo Artinian modules. Finally, in Theorem 2.10, some topological properties of the prime spectrum of pseudo Artinian modules are investigated.

2. pseudo Artinian modules

By definition, any Artinian module is pseudo Artinian, see [4, Corollary 2.4]. However, the converse is not true in general. For example, every infinite vector space over a field is pseudo Artinian which is not Artinian. So, the class of pseudo Artinian modules is more extensive than the class of Artinian modules. Now, we present other examples of pseudo Artinian modules.

Example 2.1. If R is a ring of (Krull) dimension 0 (e.g, Artinian or absolutely flat ring), then every R-module is a pseudo Artinian module. Let S be a one-dimensional integral domain and M be a S-module such that $\mathrm{Spec}_{(0)}(M) = \emptyset$. Then, M is a pseudo Artinian S-module. For instance, if S is a Dedekind domain, then every torsion S-module is pseudo Artinian.

In the next lemma, we will present some properties of pseudo Artinian modules. Recall that, the *saturation* of N with respect to a prime ideal $\mathfrak{p}$ of R, denoted by $S_{\mathfrak{p}}(N)$, is the kernel of the composite homomorphism

$$M \to M/N \to M_{\mathfrak{p}}/N_{\mathfrak{p}}$$

where the first homomorphism is the canonical homomorphism (see [6, p.69]). More precisely,

$$S_{\mathfrak{p}}(N) = \{m \in M \mid sm \in N \text{ for some } s \in R \setminus \mathfrak{p}\}.$$

Lemma 2.2.

(1) *Let R be an integral domain over which every R-module is pseudo Artinian. Then R is a field.*
(2) *Let R be an integral domain. If M is a non-primeless pseudo Artinian R-module, then either M is torsion or R is a field.*
(3) *An R-module M is a pseudo Artinian module if and only if the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is a pseudo Artinian module for every prime (or maximal) ideal $\mathfrak{p}$ of R.*
(4) *Let $\{M_i\}_{i\in I}$ be a family of R-modules. Then $\bigoplus_{i\in I} M_i$ is a pseudo Artinian R-module if and only if M_i is a pseudo Artinian R-module for each $i \in I$.*
(5) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If M' and M'' both are pseudo Artinian, then M is pseudo Artinian. Also, if M is pseudo Artinian, then so is M''.*
(6) *If $\mathrm{Supp}(M) \subseteq \mathrm{Max}(R)$, then M is pseudo Artinian.*

Proof.

(1) Let K be the field of quotients of R. Then (0) is the only (0)-prime submodule of the R-module K (see, for example [15, Theorem 1]). Since K is pseudo Artinian, we have $(0) \in \mathrm{Max}(R)$. This implies that R is a field.
(2) Suppose that M is not torsion. By [17, Lemma 4.5], $S_{(0)}(0)$ is a (0)-prime submodule of M. Since M is pseudo Artinian, (0) is a maximal ideal of R. Hence, R is a field.
(3) Let M be a pseudo Artinian R-module and Q be a $\mathfrak{q}R_{\mathfrak{p}}$-prime submodule of $M_{\mathfrak{p}}$ where $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of R. According to [15, Proposition 1], $Q \cap M$ is a $\mathfrak{q}$-prime submodule of M. So $\mathfrak{q} \in \mathrm{Max}(R)$, therefore $\mathfrak{q} = \mathfrak{p}$. Hence, $\mathfrak{q}R_{\mathfrak{p}}$ is (the unique) maximal ideal of $R_{\mathfrak{p}}$. This shows that $M_{\mathfrak{p}}$ is a pseudo Artinian module.

   Now, suppose that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is a pseudo Artinian module for every prime ideal $\mathfrak{p}$ of R and M is not a pseudo Artinian module. Then there exists a prime submodule P of M such that $\mathfrak{p} := (P : M)$ is not a maximal ideal of R. Thus, there exists a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subseteq \mathfrak{m}$. By [15, Proposition 1], P^e is a $\mathfrak{p}R_{\mathfrak{m}}$-prime submodule of the pseudo Artinian $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$. In the light of our assumption, $\mathfrak{p}R_{\mathfrak{m}}$ is the maximal ideal of $R_{\mathfrak{m}}$. Therefore, $\mathfrak{p} = \mathfrak{m}$, which is a contradiction. So, M is pseudo Artinian.
(4) Suppose that M_i is a pseudo Artinian R-module for each $i \in I$. Let P be a $\mathfrak{p}$-prime submodule of $\bigoplus_{i\in I} M_i$. Then, $M_j \nsubseteq P$ for some $j \in I$. By [22, Lemma 1.6], $P \cap M_j$ is a $\mathfrak{p}$-prime submodule of M_j. Since M_j is a pseudo Artinian R-module, $\mathfrak{p}$ is a maximal ideal of R. This implies that $\bigoplus_{i\in I} M_i$ is a pseudo Artinian R-module.

   Conversely, suppose that $\bigoplus_{i\in I} M_i$ is a pseudo Artinian R-module and let P_j be a $\mathfrak{p}_j$-prime submodule of M_j, for some $j \in I$. Then by [18, Lemma 4.6],

$$P_j \oplus \bigoplus_{j\neq i\in I} M_i \in \mathrm{Spec}_{\mathfrak{p}_j}(\bigoplus_{i\in I} M_i).$$

   Hence, $\mathfrak{p}_j$ is a maximal ideal of R. This shows that M_j is a pseudo Artinian module.
(5) We may assume that M' is a submodule of M and $M'' = M/M'$. Suppose M' and M'' are both pseudo Artinian. Let P be a $\mathfrak{p}$-prime submodule of M. If $P \cap M' = M'$, then P/M' is a prime submodule of M''. Since M'' is pseudo Artinian, $(P/M' : M'') = (P : M)$ is a maximal ideal of R, as desired. Otherwise, if $P \cap M' \neq M'$, then $P \cap M'$ is a $\mathfrak{p}$-prime submodule of M'

by [22, Lemma 1.6]. Since M' is pseudo Artinian, $\mathfrak{p}$ must be a maximal ideal of R. This implies that M is pseudo Artinian.

Now, suppose that M is pseudo Artinian. Let Q/M' be a prime submodule of M''. Then Q is a prime submodule of M, hence $(Q : M) = (Q/M' : M'')$ is a maximal ideal of R. This shows that M'' is pseudo Artinian.

(6) Let P be a prime submodule of M. Then

$$\{(P : M)\} = \mathrm{Ass}(M/P) \subseteq \mathrm{Supp}(M/P) \subseteq \mathrm{Supp}(M) \subseteq \mathrm{Max}(R).$$

This implies that $(P : M) \in \mathrm{Max}(R)$. So, M is pseudo Artinian

□

Example 2.3. We note that Lemma 2.2 enables us to construct more examples of pseudo Artinian modules. For instance, let q be a prime integer. Then $\mathbb{Z}/q\mathbb{Z}$ is a pseudo Artinian $\mathbb{Z}$-module. Hence, $\bigoplus_p \mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}$-module, where p runs through the set of all prime integers, is pseudo Artinian, by Lemma 2.2.

Remark 2.4.

(1) By Lemma 2.2, any direct sum of arbitrary family of pseudo Artinian modules is a pseudo Artinian, again. But this is not true for the case of direct product. For example, consider $L = \prod_p \mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}$-module, where p runs through the set of all prime integers. Since L is not torsion, according to the Lemma 2.2, it is not a pseudo Artinian module.

(2) Let M be a pseudo Artinian R-module. Then, a proper submodule N of M is a prime submodule of M if and only if $(N : M) \in \mathrm{Max}(R)$, by [13, Proposition 2].

(3) Every minimal prime submodule of M is of the form $\mathfrak{m}M$, for some maximal ideal $\mathfrak{m}$ of R. In particular, if R is semi-local, then M has only finitely many minimal prime submodules.

Let M be an R-module. Then, M is called *primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the *natural map* $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ for every $P \in \mathrm{Spec}(M)$, is surjective (see [18]).

Proposition 2.5. *Let M be a nonzero R-module. If M is pseudo Artinian, then* $\dim(R/\mathrm{Ann}(M)) = 0$ *in each of the following cases:*

(1) *M is primeful;*
(2) *M is free;*
(3) *M is finitely generated;*
(4) *M is faithfully flat;*
(5) *R is an integral domain and M is projective.*

Proof. It is shown in [18] that in each cases (1)-(5), for every prime ideal $\mathfrak{p}$ containing $\mathrm{Ann}(M)$, there is a prime submodule P of M such that $(P : M) = \mathfrak{p}$. By assumption, $\mathfrak{p}$ must be a maximal ideal of R. This completes the proof. □

Corollary 2.6. *Let M be a nonzero finitely generated pseudo Artinian module over a Noetherian ring R. Then M has finite length.*

We recall that, an R-module M is called *catenary* if for any prime submodules P and Q of M with $P \subsetneq Q$, all the saturated chains of the prime submodules of M starting from P and ending at Q have the same length (see [23]). Let N be a submodule of an R-module M. The *radical* of N, denoted by $\mathrm{rad}_M(N)$ or briefly $\mathrm{rad}(N)$, is defined to be the intersection of all prime submodules of M containing N. In the case where there are no such prime submodules, $\mathrm{rad}(N)$ is defined as M. If $\mathrm{rad}(N) = N$, we say that N is a *radical submodule* (see [14, 20]).

Theorem 2.7. *Let M be a pseudo Artinian R-module. The following statements holds:*

(1) *M is catenary.*
(2) *If I is an ideal of R, then* $\mathrm{rad}(IM) = \bigcap_{\mathfrak{m}\in \mathrm{Max}(R)\cap V(I)} \mathfrak{m}M$.

Proof.

(1) Consider a chain of the prime submodules $P \subsetneq Q$ of M. By definition, $\mathfrak{p} := (P :_R M)$ is a maximal ideal of R. Let N be a prime submodule of M such that $P \subseteq N \subseteq Q$. Then $(N :_R M) = \mathfrak{p}$ and N/P is a (0)-prime submodule of $R/\mathfrak{p}$-vector space M/P. Therefore, any chain of the prime submodules $P \subset N_1 \subset N_2 \subset \cdots \subset Q$ of M is a saturated chain if and only if $P/P \subset N_1/P \subset N_2 \subset \cdots \subset Q/P$ is a saturated chain of $R/\mathfrak{p}$-subspaces of M/P. Consequently, length of any saturated chain of the prime submodules of M starting from P and ending at Q is equal to $rank_{R/\mathfrak{p}}(Q/P)$.
(2) If $V^*(IM) = \emptyset$, then by [13, Proposition 4], $\mathfrak{m}M = M$ for any maximal ideal $\mathfrak{m} \supseteq I$. Hence, $\mathrm{rad}(IM) = \bigcap_{\mathfrak{m}\in \mathrm{Max}(R)\cap V(I)} \mathfrak{m}M = M$. Thus, we suppose that $V^*(IM) \neq \emptyset$. Let $P \in V^*(IM)$ be a $\mathfrak{m}$-prime submodule of M. This implies that $IM \subseteq \mathfrak{m}M \subseteq P \neq M$. Again by [13, p.63, Proposition 4], $\mathfrak{m}M$ is a prime submodule of M. So, $\mathfrak{m}M$ is a minimal element of $\mathrm{Spec}_{\mathfrak{m}}(M)$. Therefore, $\mathrm{rad}(IM) = \bigcap_{\mathfrak{m}\in \mathrm{Max}(R)\cap V(I)} \mathfrak{m}M$.

□

We conclude the paper by investigating some topological properties of the prime spectrum of pseudo Artinian modules.

Remark 2.8. Let M be an R-module. By [16, Theorem 6.1], the following statements are equivalent:

(1) $(\mathrm{Spec}(M), \tau)$ is a T_0-space;
(2) $|\mathrm{Spec}_{\mathfrak{p}}(M)| \leq 1$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$.

Remark 2.9. Let X be a topological space.

(1) Let M ba an R-module and set $\mathsf{Z}^*(M) = \{V^*(N) : N \leq M\}$. There is a topology, τ^* say, on $\mathrm{Spec}(M)$ due to $\mathsf{Z}^*(M)$ as the collection of all closed sets if and only if $\mathsf{Z}^*(M)$ is closed under finite union. When this is the case, we call the topology τ^* the *quasi-Zariski topology on* $\mathrm{Spec}(M)$ and M is called a *top* module (see [22]).
(2) X is said to be *Noetherian* if the open subsets of X satisfy the ascending chain condition. X is said to be *irreducible* if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersects ([6]). For examples of modules with Noetherian spectrum, we refer the reader to [1, 19].
(3) Let M be an R-module and Y be a subset of $\mathrm{Spec}(M)$. We will denote the intersection of all elements in Y by $\Im(Y)$ and the *closure* of Y in $\mathrm{Spec}(M)$ w.r.t the (quasi-)Zariski topology by $Cl(Y)$. By [16, Proposition 5.1], we have $V(\Im(Y)) = Cl(Y)$. An element $y \in Y$ is called a *generic point* of Y if $Y = Cl(\{y\})$.
(4) Following M. Hochster [12], we say that a topological space Y is a *spectral space* in the case where Y is homeomorphic to $\mathrm{Spec}(S)$, with the Zariski topology, for some ring S. Spectral spaces have been characterized by Hochster [12, Proposition 4] as the topological spaces Y which satisfy the following conditions: (1) Y is a T_0-space; (2) Y is quasi-compact; (3) the quasi-compact open subsets of Y are closed under finite intersections and form a basis of open sets; (4) each irreducible closed subset of Y has a generic point. For examples of modules whose prime spectrum is spectral, see [1, 16].
(5) A Noetherian space is spectral if and only if it is T_0 and every non-empty irreducible closed subspace has a generic point ([12, pp. 57-58]). We recall that if M is a top R-module, then $(\mathrm{Spec}(M), \tau^*)$ is a T_0-space and every irreducible closed subset of $\mathrm{Spec}(M)$ has a generic point (see [2, Theorem 3.3]).

Theorem 2.10. *Let M be a pseudo Artinian R-module. The following statements holds:*

(1) *If* $\sum := \{P \in \mathrm{Spec}(M) \,|\, (P : M)M \neq M\}$ *is a finite set, then* $(\mathrm{Spec}(M), \tau)$ *is a Noetherian space. Moreover,* $M/\mathrm{rad}(\boldsymbol{0})$ *is a*

Noetherian R-module if and only if $M/\mathrm{rad}(\boldsymbol{0})$ is an Artinian R-module.

(2) *Let* $(\mathrm{Spec}(M), \tau)$ *be a T_0-space. Then we have*
   (a) $\mathrm{Spec}(M) = \mathrm{Max}(M)$.
   (b) *If M is content, then M is top. Moreover, if* $\mathrm{Spec}(R)$ *is Noetherian, then (X, τ^*) is spectral.*
   (c) *If R is a one-dimensional integral domain with the Noetherian spectrum, then M is top.*

Proof.

(1) Let

$$V(N_1) \supseteq V(N_2) \supseteq \cdots$$

be a descending chain of closed subsets of $(\mathrm{Spec}(M), \tau)$. So, we have an ascending chain

$$\Im(V(N_1)) \subseteq \Im(V(N_2)) \subseteq \cdots$$

of radical submodules of M and the ascending chain of radical ideals

$$(\Im(V(N_1)) : M) \subseteq (\Im(V(N_2)) : M) \subseteq \cdots .$$

Since $\sum$ is a finite set, there exists a positive integer k such that

$$(\Im(V(N_k)) : M)M = (\Im(V(N_{k+i})) : M)M$$

for each $i = 1, 2, \cdots$. By [16, Result 3], $V(\Im(V(N_k))) = V(\Im(V(N_{k+i})))$. By Remark 2.9, $V(N_k) = V(N_{k+i})$, and so $(\mathrm{Spec}(M), \tau)$ is a Noetherian space.

For the second assertion, note that by assumption and Theorem 2.7 there are finitely many maximal ideals $\mathfrak{m}_{\lambda_1}, \ldots, \mathfrak{m}_{\lambda_t}$ such that $\mathrm{rad}(\mathbf{0}) = \mathfrak{m}_{\lambda_1}M \cap \cdots \cap \mathfrak{m}_{\lambda_t}M$. This implies that $M/\mathrm{rad}(\mathbf{0})$ is annihilated by $\mathfrak{m}_{\lambda_1} \cdots \mathfrak{m}_{\lambda_t}$. Therefore, $M/\mathrm{rad}(\mathbf{0})$ is a Noetherian R-module if and only if $M/\mathrm{rad}(\mathbf{0})$ is an Artinian R-module.

(2) (a): Clearly $\mathrm{Max}(M) \subseteq \mathrm{Spec}(M)$. Let $P \in \mathrm{Spec}(M)$. Then, there is a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{m} = (P : M)$. Suppose that L is a proper submodule of M such that $P \subseteq L$. Then $\mathfrak{m} = (P : M) = (L : M)$. By [13, Proposition 4], $\mathfrak{m}M$ and L are $\mathfrak{m}$-prime submodules of M. Since $(\mathrm{Spec}(M), \tau)$ is a T_0-space, $P = L = \mathfrak{m}M$ by Remark 2.8. Consequently, $P = \mathfrak{m}M \in \mathrm{Max}(M)$. (b) and (c) are direct consequences of part (a) and [1, Theorem 3.9].

□

Let M be an R-module and N be a submodule of M. We say that N is *j-semiprime* if N is an intersection of some prime submodules P of M such that $(P : M)$ is a maximal ideal of R.

Proposition 2.11. *Let M be a pseudo Artinian R-module. Then (X,τ) is a Noetherian (and so is quasi-compact) topological space in each of the following cases:*

1. *R satisfies ACC on j-semiprime ideals;*
2. *M satisfies the ascending chain condition on submodules of the form IM, where I is an j-semiprime ideal of R;*

Proof. (1) Let

$$V(N_1) \supseteq V(N_2) \supseteq \cdots$$

be a descending chain of closed subsets of (X,τ). Then, we have an ascending chain of j-semiprime submodules of M,

$$\Im(V(N_1)) \subseteq \Im(V(N_2)) \subseteq \cdots,$$

and the ascending chain of j-semiprime ideals,

$$(\Im(V(N_1)) : M) \subseteq (\Im(V(N_2)) : M) \subseteq \cdots.$$

Thus, there exists a positive integer k such that

$$(\Im(V(N_k)) : M)M = (\Im(V(N_{k+i})) : M)M$$

for each $i = 1, 2, \cdots$. By [16, Result 3],

$$V(\Im(V(N_k))) = V(\Im(V(N_{k+i}))).$$

By Remark 2.9, $V(N_k) = V(N_{k+i})$, and so (X,τ) is a Noetherian space.

(2) Let $V(N_1) \supseteq V(N_2) \supseteq \cdots$ be a descending chain of closed subsets of (X,τ). Then we have an ascending chain of j-semiprime submodules of M, $\Im(V(N_1)) \subseteq \Im(V(N_2)) \subseteq \cdots$, and the ascending chain of j-semiprime ideals, $(\Im(V(N_1)) : M) \subseteq (\Im(V(N_2)) : M) \subseteq \cdots$. Thus, by assumption there is a positive integer k such that $(\Im(V(N_k)) : M)M = (\Im(V(N_{k+i})) : M)M$ for each $i = 1, 2, \ldots$. By [16, Result 3], $V(\Im(V(N_k))) = V(\Im(V(N_{k+i})))$. So, by Remark 2.9, $V(N_k) = V(N_{k+i})$, and so (X,τ) is a Noetherian space.

□

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