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Improvement of the Grüss type inequalities for positive linear maps on C^*-algebras

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Abstract. Assume that A and B are unital C^*-algebras and $\varphi : A \to B$ is a unital positive linear map. We show that if B is commutative, then for all $x, y \in A$ and $\alpha, \beta \in \mathbb{C}$

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq \left[\varphi(|x^* - \alpha 1_A|^2)\right]^{\frac{1}{2}} \left[\varphi(|y - \beta 1_A|^2)\right]^{\frac{1}{2}} - |\varphi(x^* - \alpha 1_A)||\varphi(y - \beta 1_A)|.$$

Furthermore, we prove that if $z \in A$ with $|z| = 1$ and $\lambda, \mu \in \mathbb{C}$ are such that $Re(\varphi((x^* - \bar{\beta} z^*)(\alpha z - x))) \geq 0$ and $Re(\varphi((y^* - \bar{\mu} z^*)(\lambda z - y))) \geq 0$, then

$$|\varphi(x^* y) - \varphi(x^* z)\varphi(z^* y)| \leq \frac{1}{4}|\beta - \alpha||\mu - \alpha| - \left[Re(\varphi((x^* - \bar{\beta} z^*)(\alpha z - x)))\right]^{\frac{1}{2}} \left[Re(\varphi((y^* - \bar{\mu} z^*)(\lambda z - y)))\right]^{\frac{1}{2}}.$$

The presented bounds for the Grüss type inequalities on C^*-algebras improve the other ones in the literature under mild conditions. As an application, using our results, we give some inequalities in $L^\infty([a, b])$, which refine the other ones in the literature.

Keywords: C^*-algebras, Grüss type inequalities, Positive linear map.

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1. Introduction

In 1935, Grüss [13] proved the following complement of Chebyshev's inequality:

$|\frac{1}{b-a}\int_a^b fg(x)dx - \frac{1}{(b-a)}\int_a^b f(x)dx \frac{1}{(b-a)}\int_a^b g(x)dx| \leq \frac{1}{4}|(M-m)||(N-n)|,$

where f and g are real integrable functions on $[a,b]$ such that there exist $m,n,M,N \in \mathbb{R}$ for which $m \leq f(x) \leq M$ and $n \leq g(x) \leq N$ for all $x \in [a,b]$. It is well-known that the constant $\frac{1}{4}$ can not be replaced by a smaller one and is derived for $f(x) = g(x) = sgn(x-(a+b)/2)$. This inequality is known as Grüss inequality in the literature and attracted the attention of many mathematicians, for more information about Grüss inequality see [18, chapter X]. The goal of these researches is to investigate and to generalize the various types of the Grüss inequality in the different area of mathematics and to obtain improved bounds for these inequalities by assuming suitable conditions, for more details see [9, 10, 22, 23]. Moreover, Grüss type inequalities have some important applications in integral arithmetic mean, difference equations, coding theory and statistics [1, 8, 17].

Recently, Dragomir [9] generalized the Grüss inequality in the setting of inner product spaces and also Ilešević and Varošanec [14] presented some results about the Grüss type inequalities in the framework of inner product modules. Also, many mathematicians are interested to study the Grüss type inequalities for positive linear maps. In 2000, Bhatia and Davis [2], presented a reverse to the so-called Kadison inequality [15]. More precisely, for a positive unital linear map φ between C^*- algebras, they proved that

$$\varphi(x^2) - \varphi(x)^2 \leq \frac{1}{4}(M-m)$$

for all self-adjoint element x with $m \leq x \leq M$. Moreover, Bhatia and Sharma [3] proved the following extension of the above inequality:

$$\varphi(x^*x) - \varphi(x)^*\varphi(x) \leq inf_{\lambda\in\mathbb{C}}\|x-\lambda\| \tag{1.1}$$

for all x. Moslehian and Rajić [19] extended (1.1) and presented the following inequality for unital $n-$positive $(n \geq 3)$ linear map φ:

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq inf_{\lambda\in\mathbb{C}}\|x-\lambda\| inf_{\mu\in\mathbb{C}}\|y-\mu\|, \tag{1.2}$$

for all x, y.

Ghazanfari and Dragomir [11] introduced a simple formulation of the Grüss type inequality in inner product C^*-modules. They also presented some generalization of the Grüss type inequalities in inner product modules. Also, Ghazanfari [12] investigated Grüss type inequality for vector-valued functions in Hilbert C*-modules.

Very recently, Dadkhah and Moslehian [7, Theorem 3.5] for $n-$positive ($n \geq 3$) linear map $\varphi : A \to B$ between two C^*-algebras A and B presented the following refined Grüss type inequality:

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq \|\varphi(|x^* - \alpha|^2)\|^{\frac{1}{2}} \|\varphi(|y - \beta|^2)\|^{\frac{1}{2}} \qquad (1.3)$$

for all $x, y \in A$ and for all $\alpha, \beta \in \mathbb{C}$. Also they proved that the inequality (1.2) is true for all unital n-positive linear maps and

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq \frac{1}{4}|\lambda - \mu||\alpha - \beta|, \qquad (1.4)$$

for all x, y and all $\alpha, \beta \in \mathbb{C}$.

In this paper, motivated and inspired by Dadkhah and Moslehian [7], assuming a mild condition, we present two improved bounds for the left sides of the inequalities (1.2) and (1.4). As an application, we give sharpened Grüss inequalities in C^*-algebra $L^\infty([a, b])$.

2. Preliminaries

Let A and B be unital C^*-algebras. An element $a \in A$ is said to be positive and is denoted by $a \geq 0$, if a is self-adjoint and $\sigma(a) \subseteq \mathbb{R}^+$. If in addition a is also invertible, then it is called strictly positive and is denoted by $a > 0$. The set of self-adjoint and positive elements of A are denoted by A_{sa} and A^+, respectively. It is well known that $|a|^2 = a^*a$ and $Re(a) = \frac{a+a^*}{2}$. The linear map $\varphi : A \to B$ is said to be unital if $\varphi(1_A) = 1_B$. Furthermore, if $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(x^*) = \varphi(x)^*$ for all $x, y \in A$, then φ is called a $*$-homomorphism. For any positive integer n, we define $\varphi_n : M_n(A) \to M_n(B)$ by $\varphi_n((a_{ij})_{n\times n}) = (\varphi(a_{ij}))_{n\times n}$, where $M_n(A)$ denotes the C^*-algebra of all $n \times n$ matrices with entries in A. The map φ is called positive if $\varphi(a) \geq 0$ for all $a \in A^+$, and $n-$positive if φ_n is positive. Moreover, φ is called completely positive if φ is $n-$positive for all n. It worth noting that, every positive linear map is not completely positive, for example if $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is defined by $\varphi(A) = A^t$, then φ is positive but is not completely positive. Also, every $*-$homomorphism on a $*$-algebras is completely positive [4, Example II.6.9.3], but the converse is not true in general, for example if $\varphi : M_2(\mathbb{C}) \to \mathbb{C}$ is defined by $\varphi((a_{ij})_{2\times 2}) = \sum_{i=1}^{2} a_{ii}$, then φ is completely positive but φ is not a homomorphism because $\varphi(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix})^2 = 9$ and $\varphi(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2) = 5$.

It should be noted that, if B is commutative, using Theorem V.35.4 of [5], Theorem 3.9 of [21] and Theorem 3.11 of [21], we can conclude that every unital positive linear map from A to a commutative C^*-algebra B, is completely positive.

Lemma 2.1. *Assume that A is a commutative unital C^*-algebra and $a, b, c, d \in A_{sa}$. The following statements hold:*

i) *If b is strictly positive and $a \leq cb^{-1}$, then $ab \leq c$.*
ii) *If $0 \leq a \leq b$ and $0 \leq c \leq d$, then $ac \leq bd$.*
iii) $(a^2 - b^2)(c^2 - d^2) \leq (ac - bd)^2$.
iv) *If $a, b, c, d \in A^+$, then $ac + bd \leq (a^2 + b^2)^{\frac{1}{2}}(c^2 + d^2)^{\frac{1}{2}}$.*

3. Grüss type inequalities on C^*-algebras

In this section, we improve the bounds of the Grüss type inequalities for a positive linear map $\varphi : A \to B$ between two C^*-algebras. More precisely, assuming the commutativity of B, we obtain sharpened results for the Grüss type inequalities. For this purpose some elementary results have been proven.

Lemma 3.1. *Assume that A and B are unital C^*-algebras, B is commutative and $\varphi : A \to B$ is a unital positive linear map. Then*

$$|\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)|^2 \leq \left[\varphi(|x|^2) - |\varphi(z^*x)|^2\right]\left[\varphi(|y|^2) - |\varphi(z^*y)|^2\right],$$

for all $x, y, z \in A$ with $|z| = 1$.

Proof. Let $x, y, z \in A$ with $|z| = 1$. Utilizing [7, Lemma, 3.4], we deduce

$$\begin{bmatrix} \varphi(|x|^2) - |\varphi(z^*x)|^2 & \varphi(x^*y) - \varphi(z^*y)\varphi(x^*z) \\ \varphi(y^*x) - \varphi(z^*x)\varphi(y^*z) & \varphi(|y|^2) - |\varphi(z^*y)|^2 \end{bmatrix} \geq 0.$$

Without lose of generality, we may suppose that $\varphi(|y|^2) - |\varphi(z^*y)|^2 > 0$, then using [6, Lemma, 2.1], we obtain

$$\varphi(|x|^2) - |\varphi(z^*x)|^2 \geq [\varphi(x^*y) - \varphi(z^*y)\varphi(x^*z)]\left[\varphi(|y|^2) - |\varphi(z^*y)|^2\right]^{-1} \times [\varphi(y^*x) - \varphi(z^*x)\varphi(y^*z)].$$

On the other hand Lemma 2.1 implies that

$$\left[\varphi(|x|^2) - |\varphi(z^*x)|^2\right]\left[\varphi(|y|^2) - |\varphi(z^*y)|^2)\right] \geq |\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)|^2.$$

The positivity of $\varphi(|y|^2) - |\varphi(z^*y)|^2$ implies that $\varphi(|y|^2) - |\varphi(z^*y)|^2 + \dfrac{1}{n} > 0$ for all $n \in \mathbb{N}$. Also since

$$\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{n} \end{bmatrix} \geq 0,$$

we have

$$\begin{bmatrix} \varphi(|x|^2) - |\varphi(z^*x)|^2 & \varphi(x^*y) - \varphi(z^*y)\varphi(x^*z) \\ \varphi(y^*x) - \varphi(z^*x)\varphi(y^*z) & \varphi(|y|^2) - |\varphi(y)|^2 + \frac{1}{n} \end{bmatrix} \geq 0.$$

Therefore

$$\left[\varphi(|x|^2) - |\varphi(z^*x)|^2\right]\left[\varphi(|y|^2) - |\varphi(z^*y)|^2 + \frac{1}{n}\right] \geq |\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)|^2. \tag{3.1}$$

Taking the limits as $n \to \infty$ in (3.1), we deduce

$$\left[\varphi(|x|^2) - |\varphi(z^*x)|^2\right]\left[\varphi(|y|^2) - |\varphi(z^*y)|^2)\right] \geq |\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)|^2.$$

□

Theorem 3.2. *Assume that A and B are unital C^*-algebras, B is commutative and $\varphi : A \to B$ is a unital positive linear map. Then*

$$|\varphi(xy) - \varphi(xz)\varphi(z^*y)| \leq \left[\varphi(|x^* - \alpha z|^2)\right]^{\frac{1}{2}}\left[\varphi(|y - \beta z|^2)\right]^{\frac{1}{2}} \tag{3.2}$$

for all $x, y, z \in A$ with $|z| = 1$ and for all $\alpha, \beta \in \mathbb{C}$,

Proof. Let $x, z \in A$ with $|z| = 1$ and $\alpha, \beta \in \mathbb{C}$. Utilizing Lemma 3.1, we deduce

$$|\varphi(xy) - \varphi(xz)\varphi(z^*y)|^2 \leq \left[\varphi(|x^*|^2) - |\varphi(z^*x^*)|^2\right]\left[\varphi(|y|^2) - |\varphi(z^*y)|^2)\right].$$

Also, using [7, Lemma 3.1], we can derive that

$$0 \leq \varphi(|x^*|^2) - |\varphi(z^*x^*)|^2 \leq \varphi(|x^* - \alpha z|^2$$

and

$$0 \leq \varphi(|y|^2) - |\varphi(z^*y)|^2 \leq \varphi(|y - \beta z|^2),$$

so, it follows from Lemma 2.1 that

$$\left[\varphi(|x^*|^2) - |\varphi(z^*x^*)|^2\right]\left[\varphi(|y|^2) - |\varphi(z^*y)|^2\right] \leq \varphi(|x^* - \alpha z|^2)\varphi(|y - \beta z|^2).$$

Since B is commutative, by using [4, Proposition II.3.1.2], we can conclude that

$$\left[\varphi(|x^*|^2) - |\varphi(z^*x^*)|^2\right]\left[\varphi(|y|^2) - |\varphi(z^*y)|^2\right] \geq 0.$$

Applying Theorem 2.2.6 of [20], we get

$$\left[\varphi(|x^*|^2) - |\varphi(z^*x^*)|^2\right]^{\frac{1}{2}}\left[\varphi(|y|^2) - |\varphi(z^*y)|^2\right]^{\frac{1}{2}} \leq \left[\varphi(|x^* - \alpha z|^2)\right]^{\frac{1}{2}}\left[\varphi(|y - \beta z|^2)\right]^{\frac{1}{2}}.$$

Thus

$$|\varphi(xy) - \varphi(xz)\varphi(z^*y)| \leq \left[\varphi(|x^* - \alpha z|^2)\right]^{\frac{1}{2}}\left[\varphi(|y - \beta z|^2)\right]^{\frac{1}{2}}.$$

□

Theorem 3.3. *Assume that A and B are unital C^*-algebras, B is commutative and $\varphi : A \to B$ is a unital positive linear map. Then*

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq \left[\varphi(|x^* - \alpha 1_A|^2)\right]^{\frac{1}{2}} \left[\varphi(|y - \beta 1_A|^2)\right]^{\frac{1}{2}} - |\varphi(x^* - \alpha 1_A)||\varphi(y - \beta 1_A)|,$$

for all $x, y \in A$ and for all $\alpha, \beta \in \mathbb{C}$.

Proof. Letting $x, y \in A$, $\alpha, \beta \in \mathbb{C}$ and $z = 1_A$ in Lemma 3.1, we get

$$|\varphi(xy) - \varphi(x)\varphi(y)|^2 \leq \left[\varphi(|x^*|^2) - |\varphi(x^*)|^2\right] \left[\varphi(|y|^2) - |\varphi(y)|^2\right].$$

Also, it is readily seen that

$$\begin{cases} \varphi(|x^* - \alpha 1_A|^2) - |\varphi(x^* - \alpha 1_A)|^2 = \varphi(|x^*|^2) - |\varphi(x^*)|^2, \\ \varphi(|y - \beta 1_A|^2) - |\varphi(y - \beta 1_A)|^2 = \varphi(|y|^2) - |\varphi(y)|^2, \end{cases}$$

so

$$|\varphi(xy) - \varphi(x)\varphi(y)|^2 \leq \left[\varphi(|x^* - \alpha|^2) - |\varphi(x^* - \alpha)|^2\right] \left[\varphi(|y - \beta|^2) - |\varphi(y - \beta)|^2\right].$$

Thus using Lemma 2.1, we can conclude that

$$|\varphi(xy) - \varphi(x)\varphi(y)|^2 \leq \left[(\varphi(|x^* - \alpha|^2))^{\frac{1}{2}} (\varphi(|y - \beta|^2))^{\frac{1}{2}} - |\varphi(x^* - \alpha)| - |\varphi(y - \beta)|\right]^2.$$

But

$$(\varphi(|x^* - \alpha|^2))^{\frac{1}{2}} (\varphi(|y - \beta|^2))^{\frac{1}{2}} - |\varphi(x^* - \alpha)||\varphi(y - \beta)| \geq 0,$$

because it is self-adjoin and if

$$\lambda \in \sigma(\left[(\varphi(|x^* - \alpha|^2))^{\frac{1}{2}} (\varphi(|y - \beta|^2))^{\frac{1}{2}} - |\varphi(x^* - \alpha)||\varphi(y - \beta)|\right]),$$

then there exist $\gamma \in \sigma((\varphi(|x^* - \alpha|^2))^{\frac{1}{2}})$, $\mu \in \sigma((\varphi(|y - \beta|^2))^{\frac{1}{2}})$, $\zeta \in \sigma(|\varphi(x^* - \alpha)|)$ and $\eta \in \sigma(|\varphi(y - \beta)|)$, such that $\lambda = \gamma\mu - \zeta\eta$. Since $\varphi(|x^* - \alpha 1_A|^2) \geq |\varphi(x^* - \alpha 1_A)|^2$ and $\varphi(|y - \beta|^2) \geq |\varphi(y - \beta)|^2$, we can conclude that $\gamma \geq \zeta \geq 0$ and $\mu \geq \eta \geq 0$. Thus $\gamma\mu \geq \zeta\mu \geq \zeta\eta$, so $\lambda \geq 0$. Therefore

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq \left[\varphi(|x^* - \alpha 1_A|^2)\right]^{\frac{1}{2}} \left[\varphi(|y - \beta 1_A|^2)\right]^{\frac{1}{2}} - |\varphi(x^* - \alpha 1_A)||\varphi(y - \beta 1_A)|.$$

□

Now, we prove some lemmas to obtain another Grüss type inequality for a positive linear map by considering a mild conditions.

Lemma 3.4. *Suppose that A is a unital C^*-algebras and $a, b \in A$. Then*

$$|a + b|^2 \geq 4Re(a^*b).$$

Proof. Since $(a-b)^*(a-b) \geq 0$, we can conclude that $a^*a + b^*b \geq a^*b + b^*a$. Hence

$$|a+b|^2 = a^*a+b^*b+a^*b+b^*a \geq 2(a^*b+b^*a) = 4(\frac{a^*b+b^*a}{2}) = 4Re(a^*b).$$

□

Lemma 3.5. *Suppose that A and B are unital C^*-algebras and $\varphi : A \to B$ is a unital positive linear map. Then*

$$\varphi(|x|^2) - |\varphi(z^*x)|^2 = Re((\alpha - \varphi(z^*x))(\varphi(x^*z) - \bar{\beta})) - Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))),$$

for all $x, y, z \in A$ with $|z| = 1$ and for all $\alpha, \beta \in \mathbb{C}$.

Proof. Let $x, y, z \in A$ with $|z| = 1$ and suppose that $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned} &Re((\alpha - \varphi(z^*x))(\varphi(x^*z) - \bar{\beta})) - Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))) \\ &\quad = \frac{1}{2}\left[(\alpha - \varphi(z^*x)(\varphi(x^*z) - \bar{\beta}) + ((\alpha - \varphi(z^*x))(\varphi(x^*z) - \bar{\beta})^*\right] \\ &\qquad - \frac{1}{2}\left[(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))) + (\varphi((x^* - \bar{\beta}z^*)(\alpha z - x)))^*\right] \\ &\quad = \varphi(|x|^2) - |\varphi(z^*x)|^2. \end{aligned}$$

□

Using [7, corollary 3.8], we can easily conclude the following theorem.

Theorem 3.6. *Suppose that A and B are unital C^*-algebras. Assume that $\varphi : A \to B$ is a unital positive linear map. If for $x, y, z \in A$ with $|z| = 1$ and for $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\begin{cases} Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))) \geq 0, \\ \qquad \& \\ Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y))) \geq 0, \end{cases} \tag{3.3}$$

then

$$|\varphi(x^*y) - \varphi(x^*z)\varphi(z^*y)| \leq \frac{1}{4}|\alpha - \beta||\lambda - \mu|. \tag{3.4}$$

Now, utilizing the above theorem, we preset a sharper bound for the left side of (3.4), which is different the other ones in the literature.

Theorem 3.7. *Suppose that A and B are unital C^*-algebras and B is commutative. Assume that $\varphi : A \to B$ is a unital positive linear map. If for $x, y, z \in A$ with $|z| = 1$ and for $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$\begin{cases} Re(\varphi((x^* - \bar{\beta}z^*)(\alpha z - x))) \geq 0 \\ \qquad \& \\ Re(\varphi((y^* - \bar{\mu}z^*)(\lambda z - y))) \geq 0, \end{cases} \tag{3.5}$$

then

$$\begin{aligned}&|\varphi(x^*y)-\varphi(x^*z)\varphi(z^*y)|\\ &\leq \frac{1}{4}|\beta-\alpha||\lambda-\mu|\\ &\quad -\left[Re(\varphi((\alpha z-x)(x^*-\bar{\beta}z^*)))\right]^{\frac{1}{2}}\left[Re(\varphi((\lambda z-y)(y^*-\bar{\mu}z^*)))\right]^{\frac{1}{2}}.\end{aligned}$$

Proof. Assume that $x, y, z \in A$ with $|z| = 1$ and assume that $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ such that the conditions (3.5) are satisfied. Using [7, Lemma 3.1], we can derive

$$|\varphi(x^*y)-\varphi(x^*z)\varphi(z^*y)|^2 \leq \left[\varphi(|x|^2)-|\varphi(z^*x)|^2\right]\left[\varphi(|y|^2)-|\varphi(z^*y)|^2\right]. \tag{3.6}$$

Also it follows from Lemma 3.5 that

$$\begin{cases}\left[\varphi(|x|^2)-|\varphi(z^*x)|^2\right]=\\ Re((\alpha-\varphi(z^*x))(\varphi(x^*z)-\bar{\beta}))-Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x))),\\ \qquad\&\\ \left[\varphi(|y|^2)-|\varphi(z^*y)|^2\right]=\\ Re((\lambda-\varphi(z^*y))(\varphi^*(z^*y)-\bar{\mu}))-Re(\varphi((y^*-\bar{\mu}z^*)(\lambda z-y))).\end{cases}$$

Furthermore, using Lemma 3.4, we can conclude that

$$\begin{cases}Re((\alpha-\varphi(z^*x))(\varphi(x^*z)-\bar{\beta}))\leq\frac{1}{4}|\bar{\alpha}-\bar{\beta}|^2=\frac{1}{4}|\beta-\alpha|^2\\ \qquad\&\\ Re((\lambda-\varphi(z^*y))(\varphi(y^*z-\bar{\mu}))\leq\frac{1}{4}|\bar{\lambda}-\bar{\mu}|^2=\frac{1}{4}|\mu-\lambda|^2.\end{cases}$$

Thus, utilizing Lemma 2.1, we have

$$\begin{aligned}&\left[\varphi(|x|^2)-|\varphi(z^*x)|^2\right]\left[\varphi(|y|^2)-|\varphi(z^*y)|^2\right]\leq\\ &\quad\left[\frac{1}{4}|\beta-\alpha||\lambda-\mu|-\left[Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x)))\right]^{\frac{1}{2}}\left[Re(\varphi((y^*-\bar{\mu}z^*)(\lambda z-y)))\right]^{\frac{1}{2}}\right]^2,\end{aligned}$$

so, it follows from (3.6) that

$$\begin{aligned}&|\varphi(xy^*)-\varphi(x^*z)\varphi(z^*y)|^2\leq\\ &\quad\left[\frac{1}{4}|\beta-\alpha||\lambda-\mu|-\left[Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x)))\right]^{\frac{1}{2}}\left[Re(\varphi((y^*-\bar{\mu}z^*))(\lambda z-y))\right]^{\frac{1}{2}}\right]^2.\end{aligned}$$

Now we prove that

$$\frac{1}{4}|\beta-\alpha||\lambda-\mu|-\left[Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x)))\right]^{\frac{1}{2}}\left[Re(\varphi((y^*-\bar{\mu}z^*)(\lambda z-y)))\right]^{\frac{1}{2}}\geq 0.$$

Evidently,

$$\frac{1}{4}|\beta-\alpha||\lambda-\mu|-\left[Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x))\right]^{\frac{1}{2}}\left[(Re(\varphi((y^*-\bar{\mu}z^*)(\lambda z-y)))\right]^{\frac{1}{2}}$$

is self-adjoint. On the other hand, let

$$\zeta \in \sigma(\frac{1}{4}|\beta-\alpha||\lambda-\mu|-\left[Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x))\right]^{\frac{1}{2}}[(Re(\varphi((y^*-\bar{\mu}z^*)(\lambda z-y)))]^{\frac{1}{2}}),$$

so there exist $p \in \sigma(|\beta-\alpha|)$, $q \in \sigma(|\lambda-\mu|)$, $m \in \sigma(\left[Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x)))\right]^{\frac{1}{2}})$, and $n \in \sigma([Re(\varphi((y^*-\bar{\mu}z^*)(\lambda z-y)))]^{\frac{1}{2}})$ such that $\zeta = \frac{1}{4}pq-mn$. Since

$$Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x)) \leq \frac{1}{4}|\beta-\alpha|^2,$$

we can conclude that

$$\left[Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x))\right]^{\frac{1}{2}} \leq \frac{1}{2}|\beta-\alpha|,$$

so $\frac{1}{2}p-m \geq 0$. Similarly, we can prove $\frac{1}{2}q-n \geq 0$. Thus $\zeta \geq 0$.

Therefore

$$\begin{aligned}|\varphi(xy^*) &- \varphi(x^*z)\varphi(z^*y)| \\ &\leq \frac{1}{4}|\beta-\alpha||\lambda-\mu| \\ &\quad - \left[Re(\varphi((x^*-\bar{\beta}z^*)(\alpha z-x)))\right]^{\frac{1}{2}}[Re(\varphi((y^*-\bar{\mu}z^*)(\lambda z-y)))]^{\frac{1}{2}}.\end{aligned}$$

□

Corollary 3.8. *Suppose that A and B are unital C^*-algebras, B is commutative, and $\varphi : A \to B$ is a unital positive linear map. If for $x, y \in A$ and $\alpha, \beta, \lambda, \mu \in \mathbb{C}$,*

$$Re(\varphi((x-\beta)(\bar{\alpha}-x^*))) \geq 0 \quad \& \quad Re(\varphi((y^*-\bar{\mu})(\lambda-y)) \geq 0,$$

then

$$\begin{aligned}|\varphi(xy) &- \varphi(x)\varphi(y)| \\ &\leq \frac{1}{4}|\beta-\alpha||\mu-\lambda| \\ &\quad - [Re(\varphi((x-\beta))(\bar{\alpha}-x^*))]^{\frac{1}{2}}[Re(\varphi((y^*-\bar{\mu}))(\lambda-y))]^{\frac{1}{2}}.\end{aligned}$$

Proof. Letting $z = 1_A$ in Theorem 3.7, we can conclude the desired results. □

4. Grüss inequality in $L^{\infty}([a,b])$.

Now, using the obtained results in pervious section, we get refined bounded for Gruss inequality in $L^{\infty}([a,b])$.

Remark 4.1. Let $A = L^{\infty}([a,b])$ which is the usual C^*-algebra of essentially bounded functionals defined on $[a,b]$. Assume that $\varphi : A \to \mathbb{C}$ is defined by $\varphi(h) = \frac{1}{b-a}\int_a^b h(x)dx$. Let m, n, M and N be positive real numbers and $\alpha = \frac{m+M}{2}$ and $\beta = \frac{n+N}{2}$. Suppose that $f, g \in L^\infty([a,b])$ which satisfy in the following conditions

$$\begin{cases} m \le f(x) \le M, \\ n \le g(x) \le N, \end{cases}$$

for all $x \in [a,b]$. It is easily seen that φ is positive unital linear map. Using Theorem 3.3, we can derive

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b (fg)(x)dx - \frac{1}{(b-a)2}\int_a^b f(x)dx \int_a^b g(x)dx\right| \\ &\le \left[\frac{1}{b-a}\int_a^b (\left|\bar{f} - \frac{m+M}{2}1\right|^2)(x)dx\right]^{\frac{1}{2}} \left[\frac{1}{b-a}\int_a^b (\left|g - \frac{n+N}{2}\right|^2)(x)dx\right]^{\frac{1}{2}} \\ &\qquad - \left|\frac{1}{b-a}\int_a^b (\bar{f} - \frac{m+M}{2})(x)dx\right|\left|\frac{1}{b-a}\int_a^b (g - \frac{n+N}{2})(x)dx\right|. \end{aligned}$$

Therefore

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b (fg)(x)dx - \frac{1}{(b-a)2}\int_a^b f(x)dx \int_a^b g(x)dx\right| \\ &\le \frac{1}{4}(M-m)(N-n) \\ &\qquad - \left|\frac{1}{b-a}\int_a^b (f - \frac{m+M}{2})(x)dx\right| \left|\frac{1}{b-a}\int_a^b (g - \frac{n+N}{2})(x)dx\right|. \end{aligned} \tag{4.1}$$

It is readily seen that

$$Re(\varphi((\bar{f} - \bar{m})(M - f))) = \frac{1}{b-a}\int_a^b (f-m)(M-f)(x)dx \ge 0$$

and

$$Re(\varphi((\bar{g} - \bar{n})(N - g)) = \frac{1}{b-a}\int_a^b (g-n)(N-g)(x)dx \ge 0,$$

so, it follows from Corollary 3.8 that

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b (fg)(x)dx - \frac{1}{(b-a)2}\int_a^b f(x)dx \int_a^b g(x)dx\right| \\ &\le \frac{1}{4}(M-m)(N-n) \\ &\qquad - \frac{1}{b-a}\left[\int_a^b (f-m)(M-f)(x)dx\right]^{\frac{1}{2}} \left[\int_a^b (g-n)(N-g)(x)dx\right]^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

Now, we give two inequalities which are sharper than the Grüss inequality obtained by Mercer [16, Theorem 1.1].

Remark 4.2. Assume that $f, g \in L^{\infty}([a,b])$ which satisfy the conditions $0 \le f(x), g(x) \le 1$ for all $x \in [a,b]$ and $S(x) = max\{f(x), g(x)\}$ and $T(x) = min\{f(x), g(x)\}$. Suppose that $P = \{x : f(x) \ge g(x)\}$ and $Q = \{x : f(x) < g(x)\}$. It is easy to see that $\int_a^b (fg)(x)dx = \int_a^b (ST)(x)dx$. Also

$$\begin{aligned}
&\int_a^b f(x)dx \int_a^b g(x)dx - \int_a^b S(x)dx \int_a^b T(x)dx \\
&= \left[\int_P f(x)dx + \int_Q f(x)dx\right]\left[\int_P g(x)dx + \int_Q g(x)dx\right] \\
&\qquad - \left[\int_P S(x)dx + \int_Q S(x)dx\right]\left[\int_P T(x)dx + \int_Q T(x)dx\right] \\
&= \int_P (f-g)(x)dx \int_Q (g-f)(x)dx,
\end{aligned}$$

so

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b (fg)(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx \\
&\qquad + \frac{1}{(b-a)^2}\int_P (f-g)(x)dx \int_Q (g-f)(x)dx \qquad (4.3) \\
&= \frac{1}{b-a}\int_a^b (ST)(x)dx - \frac{1}{(b-a)^2}\int_a^b S(x)dx \int_a^b T(x)dx.
\end{aligned}$$

Using inequality (4.1), we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b (ST)(x)dx - \frac{1}{(b-a)^2}\int_a^b S(x)dx \int_a^b T(x)dx \\
&\qquad \le \frac{1}{4} - \frac{1}{(b-a)^2}\Big|\int_a^b (S-\frac{1}{2})(x)dx\Big|\Big|\int_a^b (T-\frac{1}{2})(x)dx\Big|
\end{aligned} \qquad (4.4)$$

and also utilizing inequality (4.2), we get

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b (ST)(x)dx - \frac{1}{(b-a)^2}\int_a^b S(x)dx \int_a^b T(x)dx \\
&\qquad \le \frac{1}{4} - \frac{1}{b-a}\left[\int_a^b (S)(1-S)(x)dx\right]^{\frac{1}{2}}\left[\int_a^b (T)(1-T)(x)dx\right]^{\frac{1}{2}}.
\end{aligned} \qquad (4.5)$$

Therefore, using the inequalities (4.3) and (4.4), we can conclude

$$\begin{aligned}\frac{1}{b-a}\int_a^b fg(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx \\ \leq \frac{1}{4} - \frac{1}{(b-a)^2}\left[\int_P (f-g)(x)dx\right]\left[\int_Q (g-f)(x)dx\right] \\ - \frac{1}{(b-a)^2}\Big|\int_a^b (S-\frac{1}{2})(x)dx\Big|\Big|\int_a^b (T-\frac{1}{2})(x)dx\Big|,\end{aligned}$$

and also utilizing the inequalities (4.3) and (4.5), we can deduce

$$\begin{aligned}\frac{1}{b-a}\int_a^b fg(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx \\ \leq \frac{1}{4} - \frac{1}{(b-a)^2}\left[\int_P (f-g)(x)dx\right]\left[\int_Q (g-f)(x)dx\right] \\ - \frac{1}{b-a}\left[\int_a^b (S)(1-S)(x)dx\right]^{\frac{1}{2}}\left[\int_a^b (T)(1-T)(x)dx\right]^{\frac{1}{2}}.\end{aligned}$$

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