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Signless Laplacian spectral determinations of some multicone graphs

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Abstract. If a clique and a regular graph are joined together the resulting graph is called a multicone graph. A graph G is said to be determined by the spectrum of its signless Laplacian matrix (DQS, for short) if every graph with the same Laplacian spectrum is isomorphic to G. It is proved that all the multicone graphs $K_r \bigtriangledown sK_t$, except for $K_r \bigtriangledown 3K_1$, are DQS, where K_r denotes a complete graph with $r \geq 1$ vertices. Consequently, by using these results we give a response to an open problem in [24].

Keywords: DQS graph; Signless Laplacian Matrix; Multicones.

2000 Mathematics subject classification: 05C50.

1. Introduction

Suppose that $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. kG means that k copies of graph G. The complement of graph G, $\overline{G}$, is a graph H such that two vertices of H are adjacent if and only if they are non-adjacent in G and also they have the same vertices. By connecting any vertex of G to each vertex of H the resulting graph is called the join of two graph G and H, i.e., $G \bigtriangledown H$. $P_{Q_{(G)}}(x) = \det(xI - Q(G))$ denotes the characteristic polynomial of signless Laplacian matrix of G, where $Q(G) = D(G) + A(G)$ and the diagonal matrix $D(G)$ and $A(G)$ respectively denote the degree matrix and the adjacency matrix of G. The signless Laplacian

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spectrum of G, $\mathrm{Spec}_Q(G) = \{[q_1]^{m_1}, [q_2]^{m_2}, ..., [q_n]^{m_n}\}$, are roots of $P_{Q_{(G)}}(x)$ and m_i denote the multiplicities of q_i. The Laplacian spectrum of a graph is defined similar and it is denoted by $P_{L_{(G)}}(x) = \det(xI - L(G))$, where $L(G) = D(G) - A(G)$. Two graphs Γ_1 and Γ_2 is called Q-cospectral(L-cospectral), if $\mathrm{Spec}_Q(\Gamma_1) = \mathrm{Spec}_Q(\Gamma_2)(\mathrm{Spec}_L(\Gamma_1) = \mathrm{Spec}_L(\Gamma_2))$. A graph G is said to be determined by the spectrum of its signless Laplacian matrix (DQS, for short) if every graph with the same Laplacian spectrum is isomorphic to G. DLS (Determined by Laplacian spectrum) is defined similar. There are many research about DLS or DQS graphs which researchers have been published so far, for example see [1–17, 25–32] and references therein. In this work, it is proved that all the graphs $K_r \bigtriangledown sK_t$, except for $K_r \bigtriangledown 3K_1$, are DQS.

2. Preliminaries

In what follows we suppose that the number of vertices, edges and triangles of graph G are denoted by n, m and t, respectively. In addition, $d_1(G) \geq d_2(G) \geq ... \geq d_n(G)$ denotes the degrees sequence of G and $q_1(G) \geq q_2(G) \geq ... \geq q_n(G)$.

Lemma 2.1 ([18]). *Consider the graph G and definite $U_l = \sum_{i=1}^{n} (q_i(G))^l$. Then*

$$U_0 = n,\ U_1 = \sum_{i=1}^{n} d_i = 2m,\ U_2 = 2m + \sum_{i=1}^{n} d_i^2 \text{ and } U_3 = 6t + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

For a connected bipartite graph G, if the orders of its vertex classes are the same, then we say G is a balanced graph and G is non-balanced, otherwise. K_1 is a non-balanced bipartite component (see [23]).

Lemma 2.2 ([23]). *If G is a graph with $n > 2$, then:*

(i) $q_2(G) \leq n - 2$.

(ii) $q_{l+1}(G) = n - 2$ $(1 \leq l < n)$ if and only if for the bipartite components of $\overline{G}$ one of the following holds:

(1) $\overline{G}$ has l balanced bipartite components,

(2) $\overline{G}$ has $l + 1$ bipartite components.

Lemma 2.3 ([24]). *If G_i is an s_i-regular graph on n_i vertices, then*

$$P_{Q(G_1 \bigtriangledown G_2)}(x) = \frac{g(x)}{(x - 2s_1 - n_2)(x - 2s_2 - n_1)} P_{Q(G_1)}(x - n_2) P_{Q(G_2)}(x - n_1),$$

where $g(x) = x^2 - (2(s_1 + s_2) + (n_1 + n_2))x + 2(2s_1s_2 + s_1n_1 + s_2n_2)$.

Lemma 2.4 ([24]). *If G_i $(i = 1, 2)$ is a graph with n_i vertices, then:*

$$P_{Q(G_1 \bigtriangledown G_2)}(x) = P_{Q(G_1)}(x-n_2) P_{Q(G_2)}(x-n_1)(1-\Gamma_{Q(G_1)}(x-n_2)\Gamma_{Q(G_2)}(x-n_1)).$$

Let M be a square matrix. For seeing the notation and terminology of $\Gamma_M(x)$ we refer the reader to [24].

Lemma 2.5 ([24])**.** *Consider r-regular graph G that is also DQS and let* $\mathrm{Spec}_Q(H) = \mathrm{Spec}_Q(G \bigtriangledown K_s)$ *and* $d_1(H) = d_2(H) = \cdots = d_s(H) = n + s - 1$*. Then* $H = G \bigtriangledown K_s$*.*

Lemma 2.6 ([19,20,22])**.** *For a graph G we always have:*

(1) $q_1(G) \leq 2d_1(G)$,

(2) $d_{n-1}(G) \geq q_{n-1}(G) - 1$ $(n \geq 2)$,

(3) $d_1(G) \leq q_1(G) - 1$ $(n \geq 2)$.

Lemma 2.7 ([12])**.** *Let n be the number of vertices of graph G. For any graph G with $q_1(G) > n - 2$, if $q_2(G) = n - 2$ and the multiplicity $n - 2$ is at least 2, then $G = T_1 \bigtriangledown T_2$ in which T_i are two graphs $(1 \leq i \leq 2)$.*

Remark 2.8 ([12]). In Lemma 2.7 the condition $q_1(G) > n-2$ is essential.The counter-example is $2K_3$.

3. Main Results

In this section it is proved that except for some classes of multicone graphs $K_r \bigtriangledown sK_t$ most of these graphs are DQS.

Lemma 3.1. *Let* $\mathrm{Spec}_Q(G) = \mathrm{Spec}_Q(K_r \bigtriangledown sK_t)$*, $r \geq 2$ and let T_i $(i = 1, 2)$ be graphs . Then* $G = T_1 \bigtriangledown T_2$*.*

Proof By Lemmas 2.3 and 2.7 the proof is completed. □

Proposition 3.2. *The signless Laplacian spectrum of $K_1 \bigtriangledown sK_t$ is:*

$$\mathrm{Spec}_Q(K_1 \bigtriangledown sK_t) = \left\{ \left[\frac{a \pm \sqrt{b^2 + 8(t-1)}}{2} \right]^1, [2t-1]^{s-1}, [t-1]^{(t-1)s} \right\},$$

where $a = st + 2t - 1$ and $b = st - 2t + 3$

Proof By Lemma 2.3 the result is sraightforwared. □

Lemma 3.3. *If* $\mathrm{Spec}_Q(H) = \mathrm{Spec}_Q(K_1 \bigtriangledown sK_t)$*, then H is either $K_1 \cup K_3$ or $K_1 \bigtriangledown sK_t$.*

Proof Consider the following two cases:

(1) H is disconnected. By the part of Introduction of [24] , if $\mathrm{Spec}_Q(H) = \mathrm{Spec}_Q(K_1 \bigtriangledown sK_1)$, then H is either $K_1 \cup K_3$ or $K_1 \bigtriangledown sK_1$. So, if H is disconnected and $t = 1$, then $H = K_1 \cup K_3$.

In the case (2) we suppose that $t \geq 2$.

(2) H is connected. Put $st + 1 = n$. By Lemma 2.6 $2st \geq q_1(K_1 \bigtriangledown sK_t)$. Also, by Lemma 2.6 and $d_1(H) + 1 \leq q_1(H) = q_1(K_1 \bigtriangledown sK_t) \leq 2st$ or $d_1(H) \leq 2st - 1$. We claim that $d_n(H) \leq t$. Suppose not and so $d_n(H) \geq t + 1$. Therefore, $d_2(H) \geq d_3(H) \geq \cdots \geq d_n(H) \geq t + 1$.

Hence $st(t+1) = st + st(t) = d_1(H) + \sum_{i=2}^{n} d_i(H) \geq d_1(H) + st(t+1)$ or $d_1(H) \leq 0$, which is impossible. Moreover, by Lemma 2.6 Proposition 3.2 $d_{n-1}(H) \geq t-2$. Hence $st + st(t) = d_1(H) + d_n(H) + \sum_{i=2}^{n-1} d_i(H) \geq d_1(H) + d_n(H) + (st-1)(t-2)$ or $d_1(H) + d_n(H) \leq 3st + t - 2$. Therefore, $d_n(H) \geq t$, otherwise $2st \geq d_1(H) \geq 3st - 1$ or $st \leq 1$, and so $t = s = 1$, a contradiction. It is easy to see that $d_n(H) = t$. By a similar argument $d_2(H) = \cdots = d_{n-1}(H) = t$ and so $d_1(H) = st$. Finally, by Lemma 2.5 the proof is completed. □

Now, we present a corollary which immediately follows from Lemma Corollary 2.2 of [24] and Lemma 3.3.

Corollary 3.4. *Note that* $K_{1,3} \bigtriangledown K_{r-1} = (3K_1 \bigtriangledown K_1) \bigtriangledown K_{r-1} = 3K_1 \bigtriangledown K_r$. *By Corollary 2.2 of [24] we get* $\mathrm{Spec}_Q(K_{1,3} \bigtriangledown K_{r-1}) = \mathrm{Spec}_Q((K_3 \cup K_1) \bigtriangledown K_{r-1})$. *This means that* $\mathrm{Spec}_Q(3K_1 \bigtriangledown K_r) = \mathrm{Spec}_Q((K_3 \cup K_1) \bigtriangledown K_{r-1})$. *However,* $(K_3 \cup K_1) \bigtriangledown K_r \neq 3K_1 \bigtriangledown K_r$, *for any natural number* r.

Theorem 3.5. *All the multicone graphs* $K_r \bigtriangledown sK_t$, *except for multicone graphs* $K_r \bigtriangledown 3K_1$, *are DQS.*

Proof We perform the induction on r. For $r = 1$, by Lemma 3.3 the proof is obvious.

The induction hypothesis: Let the problem be true for the values less than or equal to r; that is, suppose that any graph Q-cospectral with $K_r \bigtriangledown sK_t$ is DQS, for $1 \leq v \leq r$.

The induction assertion: We prove that any graph Q-cospectral with $K_{r+1} \bigtriangledown sK_t$ is DQS. To put that another way, we show that if $\mathrm{Spec}_Q(H) = \mathrm{Spec}_Q(K_{r+1} \bigtriangledown sK_t)$, then $H = K_{r+1} \bigtriangledown sK_t$.

It follows from Lemma 3.1 that $H = G_1 \bigtriangledown G_2$. Therefore, $\mathrm{Spec}_Q(H = G_1 \bigtriangledown G_2) = \mathrm{Spec}_Q(K_1 \bigtriangledown (K_r \bigtriangledown sK_t))$. Now, by Lemma 2.4 $\mathrm{Spec}_Q(G_2) = \mathrm{Spec}_Q(K_r \bigtriangledown sK_t)$ and $G_1 = K_1$. It follows from the induction hypothesis that $G_2 = K_r \bigtriangledown sK_t$ and so $H = K_1 \bigtriangledown (K_r \bigtriangledown sK_t) = K_{r+1} \bigtriangledown sK_t$, since these graphs are DQS and there is no graphs Q-cospectral non-isomorphic with them (It is easy we can consider different cases for G_1 and G_2, which all of them are DQS). It follows from the induction hypothesis that $G_2 = K_r \bigtriangledown sK_t$ and so $H = K_1 \bigtriangledown (K_r \bigtriangledown sK_t) = K_{r+1} \bigtriangledown sK_t$. The proof is complete. □

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