Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **12**(1)(2023), 204-210 (Research paper)

Some new properties of non-abelian tensor analogues of 2-auto Engel groups

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Abstract. In this paper, we study the concept of $2_{\otimes}$-auto Engel groups. Among other results, we prove that for any group G, if every element of $G\otimes Aut(G)$ is $2_{\otimes}$-Engel group, then $\left\langle (g\otimes\alpha), (g\otimes\alpha)^{g'\otimes\alpha'} \right\rangle$ is a nilpotent subgroup of class at most 2 in $G\otimes Aut(G)$, for all $g, g' \in G$ and $\alpha, \alpha' \in Aut(G)$.

Keywords: Non-abelian tensor product, auto Engel element, autocommutator subgroup.

2020 Mathematics subject classification: 20F28, 20F45.

1. Introduction and preliminaries

Let G and H be groups equipped with the actions of G on H and H on G (both from the right), written as h^g and g^h for all $g \in G$ and $h \in H$, in such a way that

$$g'^{(h^g)} = \left((g'^{g^{-1}})^h \right)^g, \qquad h'^{(g^h)} = \left((h'^{h^{-1}})^g \right)^h,$$

for all $g, g' \in G$ and $h, h' \in H$ (see [2, 3] for more information).
Clearly, a group acts on itself by conjugation. By considering the above compatibility of groups action, the non-abelian tensor product $G \otimes H$

[1]Corresponding author: darabi@esfarayen.ac.ir
Received: 03 November 2021
Revised: 15 July 2023
Accepted: 17 May 2023

is the group generated by the symbols $g \otimes h$, satisfying the following relations:

$$gg' \otimes h = \left(g^{g'} \otimes h^{g'}\right)\left(g' \otimes h\right),$$
$$g \otimes hh' = \left(g \otimes h'\right)\left(g^{h'} \otimes h^{h'}\right),$$

for all $g, g' \in G$ and $h, h' \in H$.

Let G be a group and denote Aut(G) to be the automorphisms group of G. For all elements $g_1, g_2, \ldots, g_n \in G$, we denote the commutator of g and g_1 as $[g, g_1] = g^{-1}g_1^{-1}gg_1$. The commutator of higher weight is defined inductively, as follows:

$$[g, g_1, g_2, \ldots, g_{n-1}, g_n] = [[g, g_1, g_2 \ldots, g_{n-1}], g_n].$$

If $g_1 = g_2 = \cdots = g_n$, we have

$$[g, g_1, g_1, \ldots, g_1] = [[g, g_1, g_2 \ldots, g_{n-1}], g_n] = [g,_n g_1].$$

The element $g \in G$ is called right n-Engel element, if $[g,_n g_1] = 1$, for all $g_1 \in G$. The set of all right n-Engel elements of the group G is defined as follows:

$$R_n(G) = \{g \in G : [g,_n g_1] = 1,\ forall g_1 \in G\}.$$

In [4], it is shown that $R_2(G)$ is a characteristic subgroup of G. Note that one has a similar set up for left n-Engel elements.

Moghaddam and Sadeghifard [6] introduced the concept of non-abelian tensor product $G \otimes Aut(G)$, with the action of G on $Aut(G)$ given by $\alpha^g := \alpha^{\varphi_g} = \varphi_{g^{-1}} \circ \alpha \circ \varphi_g$ for any $\alpha \in Aut(G)$ and $\varphi_g \in Inn(G)$, and the action of $Aut(G)$ on G, by $g^\alpha = \alpha(g)$.

We remind the auto-commutator subgroup (see[8], for more information)

$$K(G) =< [g, \alpha] : g \in G, \alpha \in Aut(G) > .$$

Clearly, it is a characteristic subgroup of G.

Now, one may define 2-auto Engel subgroup of G, as follows:

$$AR_2(G) = \{g \in G : [[g, \alpha], \alpha] = 1\},$$

and likewise right $2_\otimes$-auto Engel sub group of G

$$AR_2^\otimes(G) = \{g \in G : [g, \alpha] \otimes \alpha = 1_\otimes,\ \text{for all } \alpha \in Aut(G)\},$$

which is a characteristic subgroup of G and contained in $AR_2(G)$. Clearly $[\alpha, g] = [g, \alpha]^{-1}$ and so by a similar way we define the set of left $2_\otimes$-auto Engel elements of G, as follows:

$$AL_2^{\otimes}(G) = \{g \in G : [\alpha, g] \otimes \varphi_g = 1_{\otimes},\ \text{for all } \alpha \in Aut(G)\},$$

which is contained in $AL_2(G) = \{g \in G : [[\alpha, g], \varphi_g] = 1\}$. A group G is an n-auto Engel group if $[g,_n \alpha] = 1$ for all $g \in G$ and $\alpha \in Aut(G)$ and $[g,_n \alpha] = [[g,_{n-1} \alpha], \alpha]$. Similarly, the group G is called $n_{\otimes}$-auto Engel, when $[g,_{n-1} \alpha] \otimes \alpha = 1_{\otimes}$ for all $g \in G$ and $\alpha \in Aut(G)$. One can easily check that every $n_{\otimes}$-auto Engel group is also n-auto Engel (see [2]).
In this paper, among results relation to $2_{\otimes}$-auto Engel group, we prove that if the normal closure of every element in $G \otimes Aut(G)$ is a $2_{\otimes}$-Engel group, then $\left\langle (g \otimes \alpha), (g \otimes \alpha)^{g' \otimes \alpha'} \right\rangle$ is a nilpotent of class at most 2 in $G \otimes Aut(G)$, where $g, g' \in G$ and $\alpha, \alpha' \in Aut(G)$.

In the following, we list some basic and important results on non-abelian tensor product $G \otimes Aut(G)$, which will be need in the rest of the paper.

Lemma 1.1. *Let g be a right 2-auto Engel element, and let α, β, and γ be arbitrary automorphisms of a group G. Then*

- *(i)* $g^{Aut(G)} = \langle g^{\alpha} : \alpha \in Aut(G) \rangle$ *is abelian,*
- *(ii)* $[g, [\alpha, \beta]] = [g, \alpha, \beta]^2$,
- *(iii)* $[g, \alpha, \beta, \gamma]^2 = 1$.

Proof. See [9, Lemma 3.2]. □

Lemma 1.2 ([2])**.** *Let $g, g' \in G$ and let $\alpha, \beta \in Aut(G)$. The following relations are hold in $G \otimes Aut(G)$:*

- *(i)* $(g^{-1} \otimes \alpha)^g = (g \otimes \alpha)^{-1} = (g \otimes \alpha^{-1})^{\alpha}$;
- *(ii)* $(g' \otimes \beta)^{(g \otimes \alpha)} = (g' \otimes \beta)^{[g,\alpha]}$;
- *(iii)* $[g, \alpha] \otimes \beta = (g \otimes \alpha)^{-1} (g \otimes \alpha)^{\beta}$;
- *(iv)* $g' \otimes [g, \alpha] = (g \otimes \alpha)^{-g'} (g \otimes \alpha)$;
- *(v)* $[g \otimes \alpha, g' \otimes \beta] = [g, \alpha] \otimes [\varphi_{g'}, \beta]$.

If A is a subset of $Aut(G)$, then we may define the auto-tensor centralizer of A in G as follows:

$$C_G^{\otimes}(A) = \{g \in G : g \otimes \alpha = 1_{\otimes},\ \text{for all } \alpha \in A\}.$$

It is easy to check that $C_G^{\otimes}(A)$ is a subgroup of G.

Proposition 1.3 ([6])**.** *Let G be a group. Then, for all $\alpha, \beta, \gamma \in Aut(G)$, $g \in AR_2^{\otimes}(G)$, and $n \in \mathbb{Z}$, the following assertions are hold:*

- *(i)* $[g, \alpha] \otimes \beta = ([g, \beta] \otimes \alpha)^{-1}$;
- *(ii)* $[g, \alpha]^{\beta} \otimes \alpha = 1_{\otimes}$;
- *(iii)* $[g, \alpha]^n \otimes \beta = ([g, \alpha] \otimes \beta)^n$;
- *(iv)* $g^{-1} \otimes \alpha = (g \otimes \alpha)^{-1}$;

(*v*) $[g,\alpha]\otimes[\beta,\gamma]=1_{\otimes}$;
(*vi*) $g\otimes[\alpha,\beta]=([g,\alpha]\otimes\beta)^2$.

Theorem 1.4 ([6]). *For a given group G, the set of all $2_{\otimes}$-auto Engel elements is a characteristic subgroup of G.*

2. Main result

In this section, we explain some properties and a generalization of $2_{\otimes}$-auto Engel group. First, we start with some properties of two sets $AR_2^{\otimes}(G)$ and $AL_2^{\otimes}(G)$.

Lemma 2.1. *Let G be any group. Then the following conditions are hold*

(*i*) $AR_2^{\otimes}(G)\subseteq AR_2(G)$,
(*ii*) $AL_2^{\otimes}(G)\subseteq AL_2(G)$,
(*iii*) $AR_2^{\otimes}(G)\subseteq AL_2^{\otimes}(G)$.

Proof. (*i*) Let $g\in AR_2^{\otimes}(G)$ and let $\kappa:G\otimes Aut(G)\longrightarrow K(G)$ given by $\kappa(g\otimes\alpha)=[g,\alpha]$ be the autocommutator map. Then $1=\kappa([g,\alpha]\otimes\alpha)=[g,\alpha,\alpha]$, so $g\in AR_2(G)$.

(*ii*) It is proved in a similar way.

(*iii*) To prove (*iii*), suppose that $g\in AR_2^{\otimes}(G)$ and that $\alpha\in Aut(G)$. Then $1_{\otimes}=[g,\varphi_g\alpha]\otimes\varphi_g\alpha=[g,\alpha]\otimes\varphi_g\alpha=([g,\alpha]\otimes\varphi_g)^{\alpha}$. Therefore $[\alpha,g]\otimes\varphi_g=1_{\otimes}$ and so $g\in AL_2^{\otimes}(G)$. □

Theorem 2.2. *Let G be a $2_{\otimes}$-auto Engel group. Then $Aut(G)$ is nilpotent of class at most 2.*

Proof. By applying Proposition 1.3, we have

$$g\otimes[\alpha,\beta,\gamma]=([g,[\alpha,\beta]]\otimes\gamma)^2=(([g,\gamma]\otimes[\alpha,\beta])^{-1})^2=1_{\otimes}$$

for all $g\in G$ and $\alpha,\beta,\gamma\in Aut(G)$. Therefore $[\alpha,\beta,\gamma]=id_G$ and so $Aut(G)$ is nilpotent of class at most 2. □

Safa et al [9] proved that if a given group G is a 2-auto Engel group, then every maximal abelian subgroup of G is characteristic. Now, we claim that if G is a $2_{\otimes}$-auto Engel group, then every maximal abelian subgroup of G is characteristic.

Theorem 2.3. *Let G be a $2_{\otimes}$-auto Engel group. Then every maximal abelian subgroup of G that is inside the tensor center of that group of G, is characteristic.*

Proof. Let M be a maximal abelian subgroup of non-abelian group G. Since the tensor center M of G is $C_G^{\otimes}(M)=\{g\in G:g\otimes m=1_{\otimes}\text{ for all }m\in M\}$, the hypothesis implies that $M\leq C_G^{\otimes}(M)$. Now, suppose $g \in C_G^{\otimes}(M)$. Then $M\langle g\rangle$ is a subgroup of G and contained in M. By the assumption, $M = M\langle g\rangle$, so $M = C_G^{\otimes}(M)$. Now, we show that for every $\alpha \in Aut(G)$, the tensor centralizer of α in G defined by $C_G^{\otimes}(\alpha) = \{g \in G : g \otimes \alpha = 1_{\otimes}\}$, is a characteristic subgroup of G. Let β be an arbitrary automorphism of G and let $h \in C_G^{\otimes}(\alpha)$. Using Proposition 1.3 (vi), we have

$$(\beta(h) \otimes \alpha)^{\beta^{-1}} = h \otimes \alpha^{\beta^{-1}} = h \otimes \alpha[\alpha, \beta^{-1}] = ([h, \alpha] \otimes \beta^{-1})^2 = 1_{\otimes}.$$

Therefore, $\beta(h) \in C_G^{\otimes}(\alpha)$. Hence $C_G^{\otimes}(\alpha)$ is a characteristic subgroup of G. Let φ_g be the inner automorphism produced by g. Then by using the relations of the nonabelian tensor product, we have

$$M = C_G^{\otimes}(M) = \bigcap_{g\in M} C_G^{\otimes}(g) = \bigcap_{g\in M} C_G^{\otimes}(\varphi_g).$$

Hence M is a characteristic subgroup of G. □

The following proposition provides equivalent conditions for $2_{\otimes}$-auto Engel groups.

Proposition 2.4. *The following statements for a group G are equivalent:*

(i) *G is $2_{\otimes}$-auto Engel;*
(ii) $[g, \alpha] \otimes \beta = ([g, \beta] \otimes \alpha)^{-1}$ *for any* $g \in G$ *and* $\alpha, \beta \in Aut(G)$*;*
(iii) $g \otimes [\alpha, \beta] = ([g, \alpha] \otimes \beta)^2$ *for any* $g \in G$ *and* $\alpha, \beta \in Aut(G)$*.*

Proof. By Proposition 1.3, parts (i), (ii), and (iii) are equivalent. As $g \otimes [\alpha, \beta] = ([g, \alpha] \otimes \beta)^2$ for any $g \in G$ and $\alpha, \beta \in Aut(G)$, so by considering $\alpha = \beta$, the parts (ii) and (iii) gives (i). □

Proposition 2.5. *If G is a $2_{\otimes}$-auto Engel group, then $C_G^{\otimes}(\alpha) \trianglelefteq G$ for any $\alpha \in Aut(G)$.*

Proof. Let G be a $2_{\otimes}$-auto Engel group, let $h \in G$, let $\alpha \in Aut(G)$, and let $g \in C_G^{\otimes}(\alpha) \leq C_G(\alpha)$. Then $g^h \otimes \alpha = g[g, h] \otimes \alpha = [g, h] \otimes \alpha = ([g, \alpha] \otimes \phi_h)^{-1} = 1_{\otimes}$. Therefore $g^h \in C_G^{\otimes}(\alpha)$, which completes the proof. □

As $C_G^{\otimes}(\alpha)$ does not necessarily contain α, the converse of Proposition 2.5 does not hold, in general.

Lemma 2.6. *Let G be a group, let $\alpha, \beta, \gamma \in Aut(G)$, and let $g \in AR_2^{\otimes}(G)$. Then the following assertions are hold:*

(i) $[\alpha, \beta, g] \otimes \gamma = 1_{\otimes}$*;*
(ii) $[g, \alpha, \beta] \otimes [\gamma, \gamma'] = 1_{\otimes}$*.*

Proof. (i): By parts (iv) and (v) of Proposition 1.3, we have

$$[\alpha, \beta, g] \otimes \gamma = ([g, [\alpha, \beta]] \otimes \gamma)^{-[\alpha,\beta,g]} = ([g, \gamma] \otimes [\alpha, \beta])^{[\alpha,\beta,g]} = 1_{\otimes}.$$

(ii): According to [6], we know that $AR_2^{\otimes}(G)$ is always a characteristic subgroup of G. Therefore $[g,\alpha] \in AR_2^{\otimes}(G)$. Thus, part (v) of Proposition 1.3 implies that (ii) is hold. □

For a given group G, we define the $2_{\otimes}$-auto Engel margins analogues of the subgroups $2_{\otimes}$-Engel margins (see [7]) as

$$AE_1^{\otimes}(G) = \{g \in G : [gh,\alpha]\otimes\alpha = [h,\alpha]\otimes\alpha \quad \text{for all } h \in G \quad \textit{and} \quad \alpha \in Aut(G)\}.$$

Moravec [7] showed that $E_1^{\otimes}(G) = \{g \in G : [gh,h'] \otimes h' = [h,h'] \otimes h' \quad \text{for all } h,h' \in G\}$ is a characteristic subgroup of G. Now, we want to show that $AE_1^{\otimes}(G)$ is also a characteristic subgroup of G.

Theorem 2.7. *Let G be a group. Then, the set of $2_{\otimes}$-auto Engel margins is a subgroup of the group G.*

Proof. Obviously, $AE_1^{\otimes}(G)$ is a characteristic set. Now, using [6, Lemma 3.3] and the commutator properties, we have

$$[g^{-1}h,\alpha]\otimes\alpha = [g^{-1},\alpha]^{\varphi_h}[h,\alpha]\otimes\alpha = ([g^{-1},\alpha]^{\varphi_h}\otimes\alpha)^{[h,\alpha]}([h,\alpha]\otimes\alpha) = [h,\alpha]\otimes\alpha,$$

which implies that $g^{-1} \in AE_1^{\otimes}(G)$. Using [6, Lemma 3.3] and the rules of non-abelian tensor product, we obtain

$$\begin{aligned}[gah,\alpha] \otimes \alpha &= ([ga,\alpha]^{\varphi_h} \otimes \alpha)([h,\alpha] \otimes \alpha) \\ &= ([g,\alpha]^{\varphi_a\varphi_h} \otimes \alpha)^{[a,\alpha]^{\varphi_h}}([a,\alpha]^{\varphi_h} \otimes \alpha)([h,\alpha] \otimes \alpha) = [h,\alpha] \otimes \alpha,\end{aligned}$$

for all $g, a \in AE_1^{\otimes}(G)$ and $\alpha \in Aut(G)$. Therefore, this completes the proof.

□

Clearly, $AE_1^{\otimes}(G) \le AR_2^{\otimes}(G)$. Indeed the inverse of the previous relation holds when α commutes with inner automorphism φ_g for all $g \in G$. The following corollary shows the properties of nilpotency for $2_{\otimes}$-auto Engel groups.

Corollary 2.8. *For a given group G, if the normal closure of every element in $G \otimes Aut(G)$ is a $2_{\otimes}$-Engel group, then for $g, g' \in G$ and $\alpha, \alpha' \in Aut(G)$, the group $\left\langle (g \otimes \alpha), (g \otimes \alpha)^{g' \otimes \alpha'} \right\rangle$ is nilpotent of class at most 2.*

Proof. From Lemma 1.2 parts (ii) and (v), we have

$$[(g \otimes \alpha)^{(g'\otimes\alpha')}, g \otimes \alpha, (g \otimes \alpha)] = [[g,\alpha]^{[g,\alpha']}, [g,\alpha]] \otimes [g,\alpha] = 1_{\otimes}.$$

Again, by using parts (ii) and (v) of Lemma 1.2, we have

$$\begin{aligned}
[(g\otimes\alpha)^{(g'\otimes\alpha')}, & g\otimes\alpha,(g\otimes\alpha)^{(g'\otimes\alpha')}] \\
&= [[\varphi_g,\alpha]^{[g'\alpha']},[\varphi_g,\alpha]]\otimes[\varphi_g,\alpha]^{[g',\alpha']} \\
&= [[\varphi_g,\alpha],[\varphi_g,\alpha]^{[g',\alpha']^{-1}}\otimes[\varphi_g,\alpha]^{[g',\alpha']}] \\
&= ([[\varphi_g,\alpha],[\varphi_g,\alpha]^{[g',\alpha']}]\otimes[\varphi_g,\alpha]^{[g',\alpha']})^{-[[\varphi_g,\alpha],[\varphi_g,\alpha]^{[g',\alpha']}]^{-1}} \\
&= ([[\varphi_g,\alpha]^{[g',\alpha']},[\varphi_g,\alpha]]\otimes[\varphi_g,\alpha])^{-[g',\alpha'][[\varphi_g,\alpha],[\varphi_g,\alpha]^{[g',\alpha']}]^{-1}} \\
&= 1_\otimes.
\end{aligned}$$

Hence $[(g\otimes\alpha)^{(g'\otimes\alpha')},g\otimes\alpha]\in Z(\left\langle g\otimes\alpha,(g\otimes\alpha)^{(g'\otimes\alpha')}\right\rangle)$, as required. □

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