Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://doi.org/10.22080/cjms.2024.27564.1707
Caspian J Math Sci. **13**(2)(2024), 386-400 (Review Article)

A relation between Zernike polynomials and other orthogonal polynomials: A review

K. J. K. Al-Tamimi [1]
[1] Iraqi Ministry of Education, Dhi Qar Education Directorate, Research and Studies Department

Abstract. Zernike circle polynomials use for wavefront analysis because of their orthogonality over a circular. So far, different representations have been presented for these polynomials. In this paper, we introduce Zernike polynomials and their important properties. Also, we present some other orthogonal polynomials such as Bessel functions, Legendre polynomials and Jaccobi polynomials and give some important results of them. Finally, we express the relationship between Zernike orthogonal polynomials and these well-known orthogonal polynomials. For different cases, the radial polynomials in the Zernike functions are obtained using the well-known orthogonal polynomials. These relations frequently used for representing wavefront aberrations.

Keywords: Zernike polynomials, Orthogonal polynomials, Legendre polynomials, Jaccobi polynomials, Bessel functions, Wavefront analysis.

2000 Mathematics subject classification: 12E10, 33C10; Secondary 65Z05.

---

[1]Corresponding author: ka-arbi@utq.edu.iq
Received: 09 August 2024
Revised: 15 December 2023
Accepted: 23 December 2023
How to Cite: Al-Tamimi, Kamel. A relation between Zernike polynomials and other orthogonal polynomials: A review, Casp.J. Math. Sci.,**13**(2)(2024), 386-400.

1. Introduction

Orthogonal polynomials play an important role for mathematical computations. These polynomials are a useful tool for solving and interpreting typs of differential equations. Also, orthogonal polynomails are used in least square approximations of a function, Fourier series and error correcting. Some other applications of orthogonal polynomials are matching polynomials of graphs and random matrix theory. Orthogonal polynomials are used in extended Lagrange interpolation, constrained least squares approximation and Gauss quadrature for rational functions [10]. Legendre polynomials, Jaccobi polynomials, Bessel functions and Zernike polynomials are well-known orthogonal polynomials. Legendre polynomails are widely used in the determination of wave functions of electrons in the orbits of an atom [1] and in the determination of potential functions in the spherically symmetric geometry [22]. The classical Jacobi polynomials have been used extensively in mathematical analysis and practical applications [11]. Also, Bessel functions have wide applications in heat conduction in a cylindrical object, diffusion problems on a lattice, frequency-dependent friction in circular pipelines and signal processing [20]. Using generalized Bessel polynomials is obtained optimal solution of nonlinear 2D variable-order fractional optimal control problems [2] and the fractional hepatitis B treatment model is solved using generalized Bessel polynomial [4]. The generalized Bessel functions (GBF) of two-variable and one-parameter are obtained by ordinary Bessel functions which are used for solving the second partial differential equations [6].

Zernike polynomials use in fields ranging from optics, vision sciences and image processing [19]. Some other researchs are solving nonlinear system of variable-order fractional PDEs using generalized Bernoulli-Laguerre polynomials [12], solving fractional optimal control problems by generalized shifted Chebyshev polynomials [13] and finding the optimal solution of fractional hematopoietic stem cells model with generalization of Bernoulli polynomials [3]. Also, the new classes of degenerated generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials have been produced in [21] and Cesarano and et al. are given the operational results on bi-orthogonal hermite functions [7].

In this paper, we introduce some orthogonal polynomials and their properties. We also investigate the relationship between these orthogonal polynomials. Some of the important results in this paper are as follows:

- Orthogonal polynomials have many applications in science and engineering, such as wavefront analysis, random matrix theory, least square approximations and etc.
- The radial polynomials in the Zernike functions are constructed by the well-known orthogonal polynomials.
- The Pseudo-radial polynomials $\mathcal{R}_n^l(\rho)$ are obtained with radial polynomials $R_n^m(\rho)$. Therefore, Pseudo-radial polynomials are also generated by famous orthogonal polynomials.

The rest of this paper is organized as follows. In Section 2, we provide the Zernike polynomials and their properties. In Section 3, the well-known Jaccobi, Legendre and Bessel orthogonal polynomials are introduced. The relationship between the Zernike radial polynomials and the orthogonal functions are given in Section 4. Finally, some conclusions are presented in Section 5.

2. Zernike polynomials

Zernike polynomials introduced by Frits Zernike in 1934 [24]. These polynomials are a sequence of continuous orthogonal functions over a unit disk and are used to characterize the wavefront. Furthermore, the Zernike polynomials are useful in defining the magnitude and characteristics of the differences between the image formed by an optical system and the original object [15]. The Cartesian coordinates (x, y) can be converted to the polar coordinates (ρ, θ) using the functions

$$\rho = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right),$$

and conversely

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta.$$

Generally, the Zernike polynomials are produced as follows [19]

$$Z_j(\rho,\theta) = Z_n^m(\rho,\theta) = \begin{cases} \sqrt{2(n+1)}R_n^m(\rho)\cos(m\theta), & m \neq 0, \qquad j \;\; is \;\; even, \\ \sqrt{2(n+1)}R_n^m(\rho)\sin(m\theta), & m \neq 0, \qquad j \;\; is \;\; odd, \\ \sqrt{2(n+1)}R_n^m(\rho), & m = 0, \end{cases} \tag{2.1}$$

in which $R_n^m(\rho)$ is radial polynomials, n is the degree of the radial polynomials, m is the azimuthal frequency describing the repetition of the angular function, n and m are non-negative integers, $n - m \geq 0$ and $n - m$ is even. The Zernike radial polynomials $R_n^m(\rho)$ are computed in several indexing scheme.

- The `Noll` indexing scheme [5]

$$R_n^m(\rho) = \frac{1}{\Big(\dfrac{n-m}{2}\Big)!\rho^m}\left[\Big(\frac{d}{d\rho^2}\Big)^{\frac{n-m}{2}}\Big((\rho^2)^{\frac{n+m}{2}}(\rho^2-1)^{\frac{n-m}{2}}\Big)\right]$$
$$= \sum_{k=0}^{\frac{n-m}{2}} \frac{(-1)^k(n-k)!}{k!\Big(\dfrac{n+m}{2}-k\Big)!\Big(\dfrac{n-m}{2}-k\Big)!}\rho^{n-2k}. \quad (2.2)$$

- The `OSA/ANSI` indexing scheme [19]

$$R_n^{|m|}(\rho) = \sum_{k=0}^{\frac{n-|m|}{2}} \frac{(-1)^k(n-k)!}{k!\Big(\dfrac{n+|m|}{2}-k\Big)!\Big(\dfrac{n-|m|}{2}-k\Big)!}\rho^{n-2k}, \quad (2.3)$$

  where n is a non-negative integer, m is an integer, $n-|m| \geq 0$ and is even.
- `Pseudo-radial` indexing scheme [19]

$$\mathcal{R}_n^l(\rho) = \sum_{k=0}^{n-|l|} \frac{(-1)^k(2n+1-k)!}{k!(n+|l|+1-k)!(n-|l|-k)!}\rho^{n-k}, \quad (2.4)$$

  in which n is a non-negative integer, l is an integer and $n-|l| \geq 0$.

Now, we use the following relationships to convert a given value j to n and m [16]

$$n = \lfloor\sqrt{2j-1}+0.5\rfloor - 1,$$
$$m = \begin{cases} 2\Big\lfloor\dfrac{2j+1-n(n+1)}{4}\Big\rfloor, & n \ \ is \ \ even, \\ 2\Big\lfloor\dfrac{2(j+1)-n(n+1)}{4}\Big\rfloor - 1, & n \ \ is \ \ odd. \end{cases}$$

Some of the Zernike radial polynomials are plotted in Figure 1. Zernike orthogonal polynomials in Cartesian and polar coordinates for $j=1$ to $j=15$ are given in Table 1 [19].

The most important properties of Zernike orthogonal polynomials are presented in the following.

Theorem 2.1. [19] *For the Zernike radial polynomials, we have* $R_n^n(\rho) = \rho^n$ *and* $R_n^n(1) = 1$.

Theorem 2.2. [19] *Let* $Z_n^m(\rho,\theta)$ *be the Zernike orthogonal polynomials. Then*

$$\Big|Z_n^m(\rho,\theta)\Big| \leq \sqrt{\frac{2(n+1)}{1+\delta_{m0}}}.$$

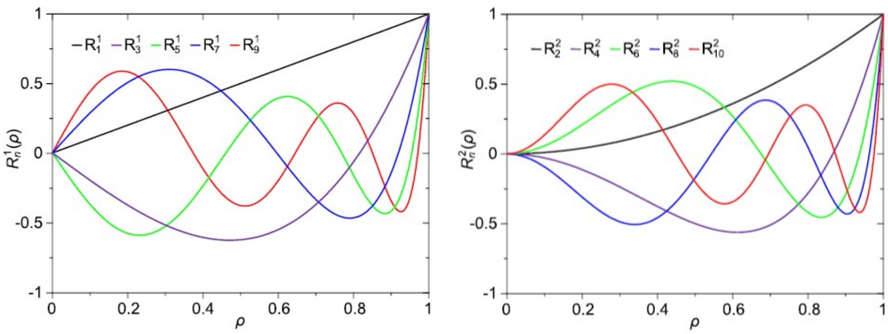

FIGURE 1. Zernike radial polynomials: (left) $m = 1$, $n = 1, 3, 5, 7, 9$ (right) $m = 2$, $n = 2, 4, 6, 8, 10$.

TABLE 1. Zernike orthonormal polynomials in `Noll` indices for $j = 1$ to $j = 15$.

j	n	m	$Z_n^m(\rho,\theta)$	$Z_n^m(x,y)$
1	0	0	1	1
2	1	1	$2\rho\cos\theta$	$2x$
3	1	1	$2\rho\sin\theta$	$2y$
4	2	0	$\sqrt{3}(2\rho^2-1)$	$\sqrt{3}\Big(2(x^2+y^2)-1\Big)$
5	2	2	$\sqrt{6}\rho^2\sin(2\theta)$	$2\sqrt{6}xy$
6	2	2	$\sqrt{6}\rho^2\cos(2\theta)$	$\sqrt{6}(x^2-y^2)$
7	3	1	$\sqrt{8}\Big(3\rho^3-2\rho\Big)\sin\theta$	$\sqrt{8}y\Big(3(x^2+y^2)-2\Big)$
8	3	1	$\sqrt{8}\Big(3\rho^3-2\rho\Big)\cos\theta$	$\sqrt{8}x\Big(3(x^2+y^2)-2\Big)$
9	3	3	$\sqrt{8}\rho^3\sin(3\theta)$	$\sqrt{8}y\Big(3x^2-y^2\Big)$
10	3	3	$\sqrt{8}\rho^3\cos(3\theta)$	$\sqrt{8}x\Big(x^2-3y^2\Big)$
11	4	0	$\sqrt{5}\Big(6\rho^4-6\rho^2+1\Big)$	$\sqrt{5}\Big(6(x^2+y^2)^2-6(x^2+y^2)+1\Big)$
12	4	2	$\sqrt{10}\Big(4\rho^4-3\rho^2\Big)\cos(2\theta)$	$\sqrt{10}(x^2-y^2)\Big(4(x^2+y^2)-3\Big)$
13	4	2	$\sqrt{10}\Big(4\rho^4-3\rho^2\Big)\sin(2\theta)$	$2\sqrt{10}xy\Big(4(x^2+y^2)-3\Big)$
14	4	4	$\sqrt{10}\rho^4\cos(4\theta)$	$\sqrt{10}\Big((x^2+y^2)^2-8x^2y^2\Big)$
15	4	4	$\sqrt{10}\rho^4\sin(4\theta)$	$4\sqrt{10}xy(x^2-y^2)$

Theorem 2.3. [19] *The Zernike radial polynomials are orthogonal*

$$\int_0^1 R_n^m(\rho)R_{n'}^m(\rho)\rho d\rho = \begin{cases} \dfrac{1}{2(n+1)}, & n = n', \\ 0, & n \neq n'. \end{cases}$$

Theorem 2.4. [19] *The Zernike polynomials are orthogonal with the weight function ρ.*

$$\int_0^1 \int_0^{2\pi} Z_n^m(\rho,\theta) Z_{n'}^{m'}(\rho,\theta) \rho d\theta d\rho = \begin{cases} \pi, & n = n', \quad m = m', \\ 0, & n \neq n', \quad m \neq m'. \end{cases}$$

Theorem 2.5. [19] *The symmetry of Zernike orthogonal polynomials can be expressed by*

$$Z_n^m(\rho,\theta) = (-1)^m Z_n^m(\rho, \theta + \pi).$$

3. Some orthogonal polynomials

In this section, we introduce the well-known orthogonal polynomials and present some the properties of them. In the next section, we prove the relationship between these polynomials and the Zernike orthogonal polynomials.

Definition 3.1. [17] *The polynomials $\{\pi_k\}$ is called a system of orthonormal polynomials respect to the inner product $(\cdot,\cdot)$ where*

$$\pi_k(x) = b_k x^k + c_k x^{k-1} + \textit{lower degree terms}, \qquad b_k > 0$$

and

$$(\pi_n, \pi_m) = \delta_{nm}, \qquad n, m \geq 0.$$

Definition 3.2. [17] *The orthogonal polynomials $\{Q_k\}$ on (a,b) with the inner product are called the classical orthogonal polynomials if their weight functions $x \longmapsto w(x)$ satisfy the differential equation*

$$\frac{d}{dx}\Big(A(x)w(x)\Big) = B(x)w(x),$$

where

$$A(x) = \begin{cases} 1 - x^2, & (a,b) = (-1,1), \\ x, & (a,b) = (0,+\infty), \\ 1, & (a,b) = (-\infty,+\infty), \end{cases}$$

and $B(x)$ is a polynomial of the first degree.

The classical orthogonal polynomials $\{Q_k\}$ on (a,b) can be specificated as follows:

- Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ for $(a,b) = (-1,1)$ and $\alpha, \beta > -1$.
- Generalized Laguerre polynomials $L_n^s(x)$ for $(a,b) = (0,+\infty)$ and $s > -1$.
- Hermite polynomials $H_n(x)$ for $(a,b) = (-\infty,+\infty)$

3.1. **Jaccobi polynomials.** For Jacobi orthogonal polynomials $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $A(x) = 1-x^2$ and $B(x) = \beta - \alpha - (\alpha+\beta+2)x$. The Jacobi orthogonal polynomials are obtained by RODRIGUES formula as [23]

$$J_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!(1-x)^{\alpha}(1+x)^{\beta}} \frac{d^n}{dx^n}\Big[(1-x)^{n+\alpha}(1+x)^{n+\beta}\Big], \quad (3.1)$$

in which $\alpha, \beta > -1$. The explicit formula to compute the Jacobi polynomials is

$$J_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha)!(n+\beta)!}{2^n} \sum_{k=0}^{n} \frac{1}{k!(n+\alpha-k)!(n-k)!(\beta+k)!}(x-1)^{n-k}(x+1)^k. \quad (3.2)$$

The Jacobi polynomials are orthogonal with the weight function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ in interval $[-1,1]$. Hence

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta} J_n^{(\alpha,\beta)}(x) J_{n'}^{(\alpha,\beta)}(x)dx = \begin{cases} \dfrac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\dfrac{(n+\alpha)!(n+\beta)!}{n!(n+\alpha+\beta)!}, & n = n', \\ 0, & n \neq n'. \end{cases} \quad (3.3)$$

Special cases of the Jacobi orthogonal polynomials are as follows [8, 9]:

- The Legendre polynomials $P_n(x)$ for $\alpha = \beta = 0$.
- The Chebyshev polynomials of the first kind $T_n(x)$ for $\alpha = \beta = -\frac{1}{2}$.
- The Chebyshev polynomials of the second kind $S_n(x)$ for $\alpha = \beta = \frac{1}{2}$.
- The Chebyshev polynomials of the third kind $U_n(x)$ for $\alpha = -\beta = -\frac{1}{2}$.
- The Chebyshev polynomials of the fourth kind $V_n(x)$ for $\alpha = -\beta = \frac{1}{2}$.

3.2. **Legendre polynomials.** Legendre polynomials, called Legendre functions of the first type, are the solutions of the Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$. These polynomials are obtained using $\alpha = \beta = 0$ in Jacobi orthogonal polynomials (3.1). Hence, the RODRIGUES formula to compute the Legendre polynomials is

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\Big[(x^2-1)^n\Big]. \quad (3.4)$$

An explicit formula to obtain the Legendre polynomials is as follows [18]

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}. \tag{3.5}$$

The Legendre polynomials are orthogonal respect to the weight function $w(x) = 1$ in $[-1, 1]$. Hence

$$\int_{-1}^{1} P_n(x) P_{n'}(x) dx = \begin{cases} \dfrac{2}{2n+1}, & n = n', \\ 0, & n \neq n'. \end{cases} \tag{3.6}$$

3.3. **Bessel functions.** The Bessel function of the first kind is defined by [5]

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}. \tag{3.7}$$

The Bessel function of the first kind satisfies in the Bessel differential equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0.$$

For the integer value n, we have

$$J_{-n}(x) = (-1)^n J_n(x).$$

4. The relations between Zernike and orthogonal polynomials

In this section, we show the relationship between the Zernike radial polynomials and the orthogonal functions.

Theorem 4.1. [23]*(The Zernike radial polynomials and the Jaccobi polynomials)*
The Zernike radial polynomials and the Jaccobi polynomials satisfy in

$$R_n^m(\rho) = (-1)^{\frac{n-m}{2}} \rho^m J_{\frac{n-m}{2}}^{(m,0)}(1 - 2\rho^2). \tag{4.1}$$

Proof. By putting $\alpha = m$, $\beta = 0$, $n = \frac{n-m}{2}$ and $x = 1 - 2\rho^2$ in (3.1), we obtain

$$\begin{aligned}
(-1)^{\frac{n-m}{2}} \rho^m J^{(m,0)}_{\frac{n-m}{2}}(1-2\rho^2) &= (-1)^{\frac{n-m}{2}} \rho^m \frac{(-1)^{\frac{n-m}{2}}}{2^{\frac{n-m}{2}}\left(\frac{n-m}{2}\right)!(2\rho^2)^m} \\
&\times \left(\frac{d}{d(1-2\rho^2)}\right)^{\frac{n-m}{2}} \left[(2\rho^2)^{\frac{n-m}{2}+m}(2-2\rho^2)^{\frac{n-m}{2}}\right] \\
&= \frac{(-1)^{n-m}}{2^{\frac{n+m}{2}}\left(\frac{n-m}{2}\right)!\rho^m} \left(\frac{d}{d(1-2\rho^2)}\right)^{\frac{n-m}{2}} \left[2^n \rho^{n+m}(1-\rho^2)^{\frac{n-m}{2}}\right] \\
&= \frac{(-1)^{\frac{3(n-m)}{2}} 2^{\frac{n-m}{2}}}{\left(\frac{n-m}{2}\right)!\rho^m} \left(\frac{d}{d(1-2\rho^2)}\right)^{\frac{n-m}{2}} \left[(\rho^2)^{\frac{n+m}{2}}(\rho^2-1)^{\frac{n-m}{2}}\right] \\
&= \frac{(-1)^{\frac{3(n-m)}{2}} 2^{\frac{n-m}{2}}}{\left(\frac{n-m}{2}\right)!\rho^m(-1)^{\frac{n-m}{2}}2^{\frac{n-m}{2}}} \left(\frac{d}{d(\rho^2-\frac{1}{2})}\right)^{\frac{n-m}{2}} \left[(\rho^2)^{\frac{n+m}{2}}(\rho^2-1)^{\frac{n-m}{2}}\right] \\
&= \frac{1}{\left(\frac{n-m}{2}\right)!\rho^m} \left[\left(\frac{d}{d\rho^2}\right)^{\frac{n-m}{2}} \left((\rho^2)^{\frac{n+m}{2}}(\rho^2-1)^{\frac{n-m}{2}}\right)\right] \\
&= R_n^m(\rho).
\end{aligned}$$

□

Therefore, the Zernike radial polynomials are a special case of the Jacobi polynomials multiplied by ρ^m.

Theorem 4.2. [18]*(The Zernike radial polynomials and the Legendre polynomials)*
For the Zernike radial polynomials and the Legendre polynomials, we have

$$R^0_{2n}(\rho) = P_n(2\rho^2 - 1). \tag{4.2}$$

Proof. By subsuitting $m = 0$ and $2n$ instead of n in (2.2), we get

$$R^0_{2n}(\rho) = \frac{1}{n!}\left[\left(\frac{d}{d\rho^2}\right)^n (\rho^2)^n(\rho^2-1)^n\right] = \frac{1}{n!}\left[\left(\frac{d}{d\rho^2}\right)^n \rho^{2n}(\rho^2-1)^n\right]. \tag{4.3}$$

Now, the RODRIGUES formula to Legendre polynomials leads to

$$\begin{aligned} P_n(2\rho^2-1) &= \frac{1}{2^n n!}\left[\left(\frac{d}{d(2\rho^2-1)}\right)^n (4\rho^4-4\rho^2)^n\right] \\ &= \frac{2^n}{n!}\left[\left(\frac{d}{d(2\rho^2-1)}\right)^n \rho^{2n}(\rho^2-1)^n\right] \\ &= \frac{2^n}{n!2^n}\left[\left(\frac{d}{d(\rho^2-\frac{1}{2})}\right)^n \rho^{2n}(\rho^2-1)^n\right] \\ &= \frac{1}{n!}\left[\left(\frac{d}{d\rho^2}\right)^n \rho^{2n}(\rho^2-1)^n\right]. \end{aligned} \tag{4.4}$$

Therefore, from (4.3) and (4.4) result in

$$R_{2n}^0(\rho) = P_n(2\rho^2-1).$$

□

For an even number n and $m = 0$, the radial polynomials are made with Legendre polynomials.

Theorem 4.3. [5]*(The Zernike radial polynomials and the Bessel functions)*
The Zernike radial polynomials and the Bessel functions satisfy in

$$\int_0^1 R_n^m(\rho)J_m(x\rho)\rho d\rho = (-1)^{\frac{n-m}{2}}\frac{J_{n+1}(x)}{x}. \tag{4.5}$$

Proof. From (2.2) and (3.7), we have

$$\begin{aligned} \int_0^1 R_n^m(\rho)J_m(x\rho)\rho d\rho &= \int_0^1 \frac{1}{\left(\frac{n-m}{2}\right)!\rho^m}\left[\left(\frac{d}{d\rho^2}\right)^{\frac{n-m}{2}}\left((\rho^2)^{\frac{n+m}{2}}(\rho^2-1)^{\frac{n-m}{2}}\right)\right] \\ &\times \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(m+k)!}\left(\frac{x\rho}{2}\right)^{m+2k}\rho d\rho \\ &= \frac{1}{\left(\frac{n-m}{2}\right)!}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{m+2k} \\ &\times \int_0^1\left[\left(\frac{d}{d\rho^2}\right)^{\frac{n-m}{2}}\left((\rho^2)^{\frac{n+m}{2}}(\rho^2-1)^{\frac{n-m}{2}}\right)\right]\rho^{2k+1}d\rho. \end{aligned}$$

By substitution $u = \rho^2$ and $du = 2\rho d\rho$, we get

$$\int_0^1 R_n^m(\rho) J_m(x\rho)\rho d\rho = \frac{1}{2\left(\frac{n-m}{2}\right)!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{m+2k} \times \int_0^1 u^k \left[\left(\frac{d}{du}\right)^{\frac{n-m}{2}} \left(u^{\frac{n+m}{2}} (u-1)^{\frac{n-m}{2}}\right)\right] du.$$

Let

$$f(k,p,q,r) = \int_0^1 u^k \left(\frac{d}{du}\right)^p u^q (u-1)^r du. \tag{4.6}$$

So

$$\int_0^1 R_n^m(\rho) J_m(x\rho)\rho d\rho = \frac{1}{2\left(\frac{n-m}{2}\right)!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{m+2k} f\left(k, \frac{n-m}{2}, \frac{n+m}{2}, \frac{n-m}{2}\right). \tag{4.7}$$

Integration by part of (4.6), gives us

$$f(k,p,q,r) = \left[u^k \left(\frac{d}{du}\right)^{p-1} u^q (u-1)^r\right]_0^1 - k \int_0^1 u^{k-1} \left(\frac{d}{du}\right)^{p-1} u^q (u-1)^r du. \tag{4.8}$$

The first term in (4.8) is zero. Hence

$$f(k,p,q,r) = -kf(k-1,p-1,q,r). \tag{4.9}$$

Now, we consider the following two cases.
CASE 1: $k \geq p$.
If we use the recursive relationship (4.9), p times, then

$$\begin{aligned} f(k,p,q,r) &= -kf(k-1,p-1,q,r) = k(k-1)f(k-2,p-2,q,r) \\ &= \cdots = (-1)^p k(k-1)\cdots(k-p+1) f(k-p,0,q,r) \\ &= \frac{(-1)^p k(k-1)\cdots(k-p+1)(k-p)(k-p-1)\cdots 1}{(k-p)(k-p-1)\cdots 1} f(k-p,0,q,r) \\ &= \frac{(-1)^{p+r} k!}{(k-p)!} \int_0^1 u^{k-p+q} (1-u)^r du. \end{aligned}$$

Finally, we will have with beta function [14]

$$f(k,p,q,r) = \frac{(-1)^{p+r} k!}{(k-p)!} \frac{\Gamma(k-p+q+1)\Gamma(r+1)}{\Gamma(k-p+q+r+2)} = \frac{(-1)^{p+r} k!(k-p+q)!r!}{(k-p)!(k-p+q+r+1)!}. \tag{4.10}$$

CASE 2: $k < p$

We use the recursive relationship (4.9), k times

$$
\begin{aligned}
f(k,p,q,r) &= -kf(k-1,p-1,q,r) = k(k-1)f(k-2,p-2,q,r) \\
&= \cdots = (-1)^k k(k-1)\cdots 1 f(0,p-k,q,r) \\
&= (-1)^k k! \int_0^1 \left(\frac{d}{du}\right)^{p-k} u^q (u-1)^r du \\
&= (-1)^k k! \left[\left(\frac{d}{du}\right)^{p-k-1} u^q (u-1)^r \right]_0^1 = 0. \qquad (4.11)
\end{aligned}
$$

By putting (4.10) and (4.11) in (4.7), we have

$$
\begin{aligned}
\int_0^1 R_n^m(\rho) J_m(x\rho)\rho d\rho &= \frac{1}{2\left(\frac{n-m}{2}\right)!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{m+2k} \\
&\times \frac{(-1)^{n-m} k! \left(k - \frac{n-m}{2} + \frac{n+m}{2}\right)! \left(\frac{n-m}{2}\right)!}{\left(k - \frac{n-m}{2}\right)! \left(k - \frac{n-m}{2} + \frac{n+m}{2} + \frac{n-m}{2} + 1\right)!} \\
&= \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^{n-m+k}}{\left(k - \frac{n-m}{2}\right)! \left(k + \frac{n+m}{2} + 1\right)!} \left(\frac{x}{2}\right)^{m+2k+1}.
\end{aligned}
$$

By changing index $l = k - \frac{n-m}{2}$, we obtain

$$
\begin{aligned}
\int_0^1 R_n^m(\rho) J_m(x\rho)\rho d\rho &= \frac{1}{x} \sum_{l=0}^{\infty} \frac{(-1)^{n-m+l+\frac{n-m}{2}}}{l! \left(l + \frac{n-m}{2} + \frac{n+m}{2} + 1\right)!} \left(\frac{x}{2}\right)^{m+2l+n-m+1} \\
&= \frac{(-1)^{\frac{3(n-m)}{2}}}{x} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(n+l+1)!} \left(\frac{x}{2}\right)^{n+2l+1} \\
&= (-1)^{\frac{n-m}{2}} \frac{J_{n+1}(x)}{x}.
\end{aligned}
$$

□

Theorem 4.4. [19]*(The Zernike radial polynomials and the pseudo Zernike radial polynomials)*
For the Zernike radial polynomials in `OSA/ANSI` *indexing scheme and the* `pseudo radial` *polynomials, we have*

$$\frac{1}{\rho} R_{2n+1}^{2l+1}(\rho) = \mathcal{R}_n^l(\rho^2). \qquad (4.12)$$

Proof. Using (2.3), we have

$$\frac{1}{\rho}R_{2n+1}^{2l+1}(\rho)=\frac{1}{\rho}\sum_{k=0}^{\frac{2n+1-2|l|-1}{2}}\frac{(-1)^k(2n+1-k)!}{k!\Big(\frac{2n+1+2|l|+1}{2}-k\Big)!\Big(\frac{2n+1-2|l|-1}{2}-k\Big)!}\rho^{2n+1-2k}$$

$$=\sum_{k=0}^{n-|l|}\frac{(-1)^k(2n+1-k)!}{k!\big(n+|l|+1-k\big)!\big(n-|l|-k\big)!}\rho^{2n-2k}. \tag{4.13}$$

Now, (2.4) gives us

$$\mathcal{R}_n^l(\rho^2)=\sum_{k=0}^{n-|l|}\frac{(-1)^k(2n+1-k)!}{k!\big(n+|l|+1-k\big)!\big(n-|l|-k\big)!}\rho^{2n-2k}. \tag{4.14}$$

Finally, from (4.13) and (4.14), we obtain

$$\frac{1}{\rho}R_{2n+1}^{2l+1}(\rho)=\mathcal{R}_n^l(\rho^2).$$

□

Theorem 4.4 show that for odd numbers m and n, radial polynomials are calculated with pseudo Zernike radial polynomials.

5. Conclusion

Orthogonal polynomials have many applications in science, medical and engineering. In this paper, we proved the relations between the Zernike radial polynomials with the Jaccobi polynomials, the Legendre polynomials, the Bessel functions and the pseudo Zernike radial polynomials. The radial polynomials for Zernike functions are computed using the well-known orthogonal polynomials. The obtained results are used in solving differential equations. These polynomials have several applications in engineering and medical sciences. In the future, we can investigate the relationship between the radial Zernike polynomials and other orthogonal polynomials.

References

[1] R. E. Attar, Special functions and orthogonal polynomials, *Lulu Press, Morrisville NC*, 2006.

[2] Z. Avazzadeh, H. Hassani, A. Bayati Eshkaftaki and P. Agarwal, Optimal solution of nonlinear 2D variable-order fractional optimal control problems using generalized Bessel polynomials, *Journal of Vibration and Control*, (2024) doi:10.1177/10775463241227475.

[3] Z. Avazzadeh, H. Hassani, M. J. Ebadi, A. Bayati Eshkaftaki, R. Katani and A. Rezvani, Generalization of Bernoulli polynomials to find optimal solution of fractional hematopoietic stem cells model, *Physica Scripta*, **99(8)** (2024), 085015.

[4] Z. Avazzadeh, H. Hassani, A. B. Eshkaftaki, M. J. Ebadi, M. K. Hosseini Asl, P. Agarwal, S. Mehrabi and M. Sh. Dahaghin, An efficient algorithm for solving the fractional hepatitis B treatment model using generalized Bessel polynomial, *Iranian Journal of Science*, **47** (2023), 1649-1664.

[5] M. Born and E. Wolf, Principles of Optics, *Cambridge: Cambridge University Press*, 1999.

[6] C. Cesarano and D. Assante, A note on generalized Bessel functions, *International Journal of Mathematical Models and Methods in Applied Sciences*, **7** (2013), 625–629.

[7] C. Cesarano, C. Fornaro and L. Vazquez, Operational results on bi-orthogonal hermite function, *Acta Mathematica Universitatis Comenianae*, **1** (2016) 43–68.

[8] G. Dattoli, S. Lorenzutta, P. E. Ricci and C. Cesarano, On a family of hybrid polynomials, *Integral Transforms and Special Functions*, **15(6)** (2004), 485–490.

[9] G. Dattoli, H. M Srivastava and C. Cesarano, The Laguerre and Legendre polynomials from an operational point of view, *Applied Mathematics and Computation*, **124(1)** (2001), 117–127.

[10] W. Gautschi, Orthogonal polynomials: applications and computation, *Acta Numerica*, (1996), 45–119.

[11] B. Guo and L. Wang, Jacobi approximations and Jacobi–Gauss–type interpolations in nonuniformly weighted Sobolev spaces, *Journal of Approximation Theory*, **128** (2004), 1–41.

[12] H. Hassani, Z. Avazzadeh, P. Agarwal, M. J. Ebadi and A. Bayati Eshkaftaki, Generalized Bernoulli-Laguerre polynomials: applications in coupled nonlinear system of variable-order fractional PDEs, Journal of Optimization Theory and Applications, **200(1)** (2024), 371–393.

[13] H. Hassani, J. A. Tenreiro Machado and E. Naraghirad, Generalized shifted Chebyshev polynomials for fractional optimal control problems, *Communications in Nonlinear Science and Numerical Simulation*, **75** (2019), 50–61.

[14] F. B. Hildebrand, *Introduction to Numerical Analysis*, 2nd Edition, McGraw-Hill, New York, 1994.

[15] V. Lakshminarayanana and A. Fleck, *Zernike polynomials: a guide*, Journal of Modern Optics, *58(7)* (2011), 545–561.

[16] V. N. Mahajan, Zernike polynomial and wavefront fitting, *Optical Shop Testing, Third Edition*, 2006.

[17] G. V. Milovanovic, Orthogonal polynomial systems and some applications, *Inner Product Spaces and Applications, by T. M. Rassias, (Addison Wesley Longman, Harlow)*, **376** (1997), 115–182.

[18] B. R. A. Nijboer, The diffraction theory of optical aberrations, *Phvsica XlII*, **10** (1947), 605–620.

[19] K. Niu and C. Tian, Zernike polynomials and their applications, *Journal of Optics*, **24** (2022), 1–54.

[20] K. Parand and M. Nikarya, Application of Bessel functions for solving differential and integro–differential equations of the fractional order, *Applied Mathematical Modelling*, **38(15-16)** (2014), 4137–4147.

[21] W. Ramırez and C. Cesarano, Some new classes of degenerated generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, *Karpatski mathematticni publikacii*, **14** (2022), 354–363.

[22] P. J. Scanlon, An alternative form for the Legendre polynomial expansion coefficients, *Journal of Computational Physics*, **69(2)** (1987), 482–486.

[23] E. W. Weisstein, CRC concise encyclopedia of mathematics, *A CRC Press Company New York*, 2003.

[24] F. Zernike, Beugungstheorie des schneidenver-fahrens und seiner verbesserten form, der phasenkontrastmethode, *Physica*, **1(7-12)** (1934), 689–704.