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On the Fourier analysis of quasi-inner product C^*-module-valued maps on LCA groups

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ABSTRACT. Given a locally compact abelian group G and a quasi-inner product A-module (M,ϕ) with ϕ a positive sesquilinear map on M, we introduce the Fourier transform for the Bochner integrable M-valued functions on G and investigate some of its properties including a Parseval type equality.

Keywords: locally compact abelian group, Fourier transform, quasi-inner product C^*-module.

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1. INTRODUCTION

The Fourier transform is an essential tool in mathematics. It gives a representation in the frequency domain of a function expressed in the time or space domain. It forms the core of harmonic analysis and can be applied to a diverse set of objects, from classical scalar valued functions to generalized distributions. Its extensions to structure

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such as abelian groups provide enhanced applications in differential equations theory, signal processing and mathematical physics, as well as a powerful framework for studying periodic and osciallatory phenomena.
Besides, positive sesquilinear maps on C^*-modules generalize the notion of inner products in Hilbert spaces. They are crucial in functional analysis and operator algebras theory, providing a suitable toolkit for studying structures similar to those of Hilbert spaces but in a more general context involving C^*-algebras. Bellomonte et al. introduced the quasi-inner product defined from a positive sesquilinear A-valued map and generalized GNS-construction for a quasi *-algebra [1]. In this framework, the linearity of the inner product with respect to the C^*-algebra does not hold. It is due to pertubations or additionnal terms in the inner product. The concept extends the notion of A-valued inner product and is useful in operator algebras theory, noncommutative geometry. For instance, in the study of physical systems, it catches the interactions or external influences lost in the standard inner product. This paper addresses the goal to combine the harmonic analysis and the notion of the quasi-inner product C^*-module in investigating some results of [9].
The structure of the paper is as follows. The Section 2 reviews some details on C*-modules with positive sesquilinear map not necessarely faithful and the characters of locally compact abelian groups. We outline our main results in Section 3.

2. Preliminaries

2.1. **Quasi-inner product A-modules.** Let A be a C^*-algebra with norm denoted $\|\cdot\|_A$ and involution denoted $*$. An element a of A is called self-adjoint if $a = a^*$ and positive if there is $b \in A$ such that $a = b^*b$. Let A^+ be the subset of positive elements of A. Let M be a right A-module. A map $\phi : M \times M \to A$ is said to be positive sesquilinear if the following statments hold for all $x, y, z \in M$ and $\alpha, \beta \in \mathbb{C}$:

(i) $\phi(x,x) \in A^+$,
(ii) $\phi(x, \alpha y + \beta z) = \alpha\phi(x, y) + \beta\phi(x, z)$,
(iii) $\phi(x, y) = \phi(y, x)^*$.

The map ϕ is called faithful if $\phi(x,x) = 0 \Rightarrow x = 0$ and A-linear if $\phi(x, ya) = \phi(x, y)a$, for all $x, y \in M$ and $a \in A$. The map ϕ is called a quasi-inner product if it is faithful, positive sesquilinear A-valued map such that

$$\|\phi(xa, xa)\|_A \leqslant \|\phi(x,x)\|_A \|a\|_A^2. \tag{2.1}$$

The couple (M,ϕ) consisting of a right A-module and a quasi-inner product is called a quasi-inner product A-module or a quasi-inner product C^*-module. Let $x \in M$, put

$$\|x\|_\phi = \|\phi(x,x)\|_A^{\frac{1}{2}}. \tag{2.2}$$

The map $x \mapsto \|x\|_\phi$ defines a quasi-norm on M, that is to say:

(i) $\forall x \in M$, $\|x\|_\phi \geqslant 0$ and $\|x\|_\phi = 0 \Leftrightarrow x = 0$,
(ii) $\forall x \in M, \forall \alpha \in \mathbb{C}$, $\|\alpha x\|_\phi = |\alpha| \|x\|_\phi$,

(iii) $\forall x, y \in M$, $\|x+y\|_\phi \leqslant \sqrt{2}(\|x\|_\phi + \|y\|_\phi)$.

The following properties hold for $\|\cdot\|_\phi$. For all $x, y \in M$ and $a \in A$:

(i) $\|xa\|_\phi \leqslant \|x\|_\phi \|a\|_A$,
(ii) $\|\phi(x,y)\|_A \leqslant 2\|x\|_\phi \|y\|_\phi$,
(iii) $\|x+y\|_\phi \leqslant \sqrt{2}(\|x\|_\phi + \|y\|_\phi)$,
(iv) If ϕ is A-linear, then $\phi(y,x)\phi(x,y) \leqslant \|\phi(x,x)\|_A \phi(y,y)$.

If ϕ is a faithful positive sesquilinear A-valued map on M, then $\|\cdot\|_\phi$ is a norm on M. More details on quasi-inner C^*-modules can be found in [1].

Example 2.1. (i) Let A be a unital C^*-algebra with unit 1_A and $M = \mathbb{C} \oplus A$ be the direct sum of the complex numbers $\mathbb{C}$ and A. The action of A on M is defined by

$$(z+a)\cdot b = z + ab. \tag{2.3}$$

The map

$$\phi : M \times M \to A, \quad (u+a, v+b) \mapsto uv1_A + a^*b \tag{2.4}$$

is positive definite, conjugate symmetric and $\mathbb{C}$-linear but not A-linear. It is a quasi-inner product on M.

(ii) Consider $M_2(\mathbb{C})$, the algebra of 2×2 matrices with complex entries. Set $M = A = M_2(\mathbb{C})$. Let u be a unitary element of $M_2(\mathbb{C})$. The set M is an A-module where the action of A on M is defined for all $x \in M$ and $a \in A$ by

$$x \cdot a = x(uau^*). \tag{2.5}$$

Denote $Tr(x)$ the trace of $x \in M_2(\mathbb{C})$ and I_2 the unit matrix of $M_2(\mathbb{C})$. The map

$$\phi : M \times M \to A, \quad (x,y) \mapsto Tr(x^*y)I_2 + xy^* - x^*y \tag{2.6}$$

is a quasi-inner product on the A-module M.

2.2. **LCA groups, Pontryagin duality and Fourier transform.** Let X be a locally compact space with a Radon measure μ. For any $p > 0$, $L^p(X,M)$ denotes the space of Bochner p-integrable M-valued functions on X [4]. It is known that the map

$$\|\cdot\|_p : L^p(X,M) \to \mathbb{R}_+,\ f \mapsto \|f\|_p = \left(\int_X \|f(u)\|_\phi^p d\mu(u)\right)^{\frac{1}{p}} \tag{2.7}$$

is a norm on $L^p(X,M)$.

Let us denote by $C_0(X,M)$ (resp. $C_c(X,M)$) the space of continuous M-valued functions on X which vanish at infinity (resp. with compact support in X). Clearly, we have $C_c(X,M) \subset C_0(X,M)$. They are both equipped with the norm of uniform convergence

$$\|g\|_\infty = \sup_{u\in X} \|g(u)\|_\phi. \tag{2.8}$$

Let $f, g \in L^2(X, M)$. Set

$$\varphi(f,g) = \int_X \phi(f(u), g(u))\,d\mu(u). \tag{2.9}$$

Moreover, for $f \in L^1(X, M)$ and $a \in A$, define the map $f \cdot a$ as follows:

$$f \cdot a : X \to M, \quad u \mapsto f(u)a. \tag{2.10}$$

Let G be a locally compact abelian group which is assumed to be Hausdorff in the whole paper [5]. Let us denote by λ its Haar measure. We call character of G, a continuous group morphism $\xi : G \to \mathbb{T}$ where $\mathbb{T}$ is the unit circle. Let $\widehat{G}$ denote the group of characters of G. It is called the dual group of G and is an abelian and Hausdorff group [3, 5] with respect to the point-wise product

$$\forall \xi, \chi \in \widehat{G}, (\xi\chi)(u) = \xi(u)\chi(u),\ u \in G. \tag{2.11}$$

The inverse of character $\xi \in \widehat{G}$ is defined by

$$\xi^{-1}(u) = (\xi(u))^{-1} = \overline{\xi(u)},\ u \in G. \tag{2.12}$$

There exists a Haar measure, denoted by Λ, on $\widehat{G}$ since, it is a locally compact group under the compact-open topology [5, 6, 8]. Again, let us denote by $\widehat{\widehat{G}}$ the dual group of $\widehat{G}$. By the Pontryagin duality [3], there exists a topological and canonical isomorphism of groups between G and its bidual $\widehat{\widehat{G}}$ defined by

$$\delta : G \to \widehat{\widehat{G}}, \quad u \mapsto \delta_u \tag{2.13}$$

where

$$\delta_u : \widehat{G} \to \mathbb{C}, \quad \xi \mapsto \delta_u(\xi) = \xi(u). \tag{2.14}$$

From [2], for all $\xi, \chi \in \widehat{G}$, we have

$$\int_G \xi(u)\overline{\chi(u)}\,d\lambda(u) = \delta_{\xi,\chi} \tag{2.15}$$

and for all $u, v \in G$, we have

$$\int_{\widehat{G}} \delta_u(\xi)\overline{\delta_v(\xi)}\,d\Lambda(\xi) = \delta_{u,v} \tag{2.16}$$

where δ denotes the Kronecker delta.
Assume that K is a compact subset of G and let r be a positive real number. Set

$$U_r = \{z \in \mathbb{C} : |1 - z| < r\} \tag{2.17}$$

and

$$N_K(r) = \{\xi \in \widehat{G} : \forall u \in K, \xi(u) \in U_r\}. \tag{2.18}$$

From [5, 7], the collection $\{N_K(r) : K \text{ compact in } G \text{ and } r > 0\}$ can be taken as a basis of neighbourhoods of the unit character of $\widehat{G}$. It defines the compact-open topology

on $\widehat{G}$.
Let $f \in L^1(G,M)$. Let us define the Fourier transform of f by

$$\hat{f}(\xi) = \int_G f(u)\overline{\xi(u)}d\lambda(u),\ \xi \in \widehat{G}. \tag{2.19}$$

For $g \in L^1(\widehat{G},M) \cap L^2(\widehat{G},M)$, the inverse Fourier transform is given by

$$\breve{g}(u) = \int_{\widehat{G}} g(\xi)\xi(u)d\Lambda(\xi),\ u \in G. \tag{2.20}$$

3. Main results

Proposition 3.1. *Let X be a locally compact space with a Radon measure μ and let (M,ϕ) be a quasi-inner product C^*-module. The space $L^2(X,M)$ with respect to φ is a quasi-inner product C^*-module.*

Proof. (1) The map φ is well defined since for all $f,g \in L^2(X,M)$, we have

$$\begin{aligned}\int_X \|\phi(f(u),g(u))\|_A d\mu(u) &\leqslant 2\int_X \|f(u)\|_\phi \|g(u)\|_\phi d\mu(u)\\ &\leqslant 2\left(\int_X \|f(u)\|_\phi^2 d\mu(u)\right)^{\frac{1}{2}}\left(\int_X \|g(u)\|_\phi^2 d\mu(u)\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

(2) Straightforward computations show that φ is positive and $\mathbb{C}$-linear in the second variable, and for all $f,g \in L^2(X,M)$, $\varphi(f,g) = \varphi(g,f)^*$.
(3) The set $L^2(X,M)$ is an A-module. Indeed, let $f,g \in L^2(X,M)$ and $a \in A$. We have

$$\begin{aligned}\int_X \|(f+g)(u)\|_\phi^2 d\mu(u) &= \int_X \|\phi(f(u)+g(u),f(u)+g(u))\|_A d\mu(u)\\ &\leqslant \int_X \|\phi(f(u),f(u))\|_A d\mu(u) + \int_X \|\phi(f(u),g(u))\|_A d\mu(u)\\ &+ \int_X \|\phi(g(u),f(u))\|_A d\mu(u) + \int_X \|\phi(g(u),g(u))\|_A d\mu(u)\\ &\leqslant \int_X \|f(u)\|_\phi^2 d\mu(u) + \int_X \|f(u)\|_\phi^2 d\mu(u)\\ &+ 4\left(\int_X \|f(u)\|_\phi^2 d\mu(u)\right)^{\frac{1}{2}}\left(\int_X \|g(u)\|_\phi^2 d\mu(u)\right)^{\frac{1}{2}}\\ &\leqslant 2\left[\left(\int_X \|f(u)\|_\phi^2 d\mu(u)\right)^{\frac{1}{2}} + \left(\int_X \|g(u)\|_\phi^2 d\mu(u)\right)^{\frac{1}{2}}\right]^2 < \infty.\end{aligned}$$

Hence, $f+g \in L^2(X,M)$ and $\|f+g\|_2 \leqslant \sqrt{2}\left(\|f\|_2 + \|g\|_2\right)$.

Moreover,

$$\begin{aligned}\int_X \|(f\cdot a)(u)\|_\phi^2 d\mu(u) &= \int_X \|f(u)a\|_\phi^2 d\mu(u)\\ &= \int_X \|\phi(f(u)a, f(u)a)\|_A d\mu(u)\\ &\leqslant \int_X \|\phi(f(u), f(u))\|_A \|a\|_A^2 d\mu(u)\\ &= \|a\|_A^2 \int_X \|f(u)\|_\phi^2 d\mu(u) < \infty.\end{aligned}$$

Hence, $f\cdot a \in L^2(X, M)$ and $\|f\cdot a\|_2 \leqslant \|a\|_A \|f\|_2$. One can easily check that the action is compatible with the operations on A. □

Let us denote by Φ and Ψ the respective A-valued quasi-inner products on $L^2(G, M)$ and $L^2(\widehat{G}, M)$ with respect to Haar measures λ and Λ respectively. The spaces $L^2(G, M)$ and $L^2(\widehat{G}, M)$ are quasi-inner product A-modules.

Proposition 3.2. *Let G be a locally compact abelian group and let (M, ϕ) be a quasi-inner product A-module. If $f \in L^1(G, M)$, then $\sup\limits_{\xi\in\widehat{G}} \|\hat{f}(\xi)\|_\phi \leqslant ||f||_1$ and $\hat{f} \in C_0(\widehat{G}, M)$.*

Proof. (i) The estimation $\sup\limits_{\xi\in\widehat{G}} \|\hat{f}(\xi)\|_\phi \leqslant ||f||_1$ is immediate.

(ii) Let $g \in C_c(G, M)$, the map $\hat{g}$ is uniformly continuous on $\widehat{G}$. Indeed, let $\epsilon > 0$ and $\xi, \chi \in \widehat{G}$. From the fact that, the Haar measure λ is regular, there is a compact subset K of G such that $|\lambda|(G\smallsetminus K) < \dfrac{\epsilon}{4(1+||g||_\infty)}$. We have

$$\|\hat{g}(\xi)-\hat{g}(\chi)\|_\phi \leqslant \int_{G\smallsetminus K} \|g(u)\|_\phi |\xi(u)-\chi(u)| d\lambda(u) + \int_K \|g(u)\|_\phi |1-\xi^{-1}(u)\chi(u)| d\lambda(u).$$

In first, we have

$$\int_{G\smallsetminus K} \|g(u)\|_\phi |\xi(u)-\chi(u)| d\lambda(u) \leqslant 2\int_{G\smallsetminus K} \|g(u)\|_\phi d\lambda(u) \leqslant 2\|g\|_\infty |\lambda|(G\smallsetminus K) \leqslant \frac{\epsilon}{2}.$$

Secondly, let $\xi^{-1}\chi \in N_k\left(\dfrac{\epsilon}{2(1+\|g\|_1)}\right)$. We have

$$\int_K \|g(u)\|_\phi |1-\xi^{-1}(u)\chi(u)| d\lambda(u) \leqslant \frac{\epsilon}{2(1+\|g\|_1)} \int_K \|g(u)\|_\phi d\lambda(u) < \frac{\epsilon}{2}.$$

Therefore, for all $\xi, \chi \in \widehat{G}$ such that $\xi^{-1}\chi \in N_k\left(\dfrac{\epsilon}{2(1+\|g\|_1)}\right)$, we have $\|\hat{g}(\xi)-\hat{g}(\chi)\|_\phi < \epsilon$. Now, let $f \in L^1(G, M)$. The map $\hat{f}$ is uniformly continuous on $\widehat{G}$. In fact, given that $C_c(G,M)$ is dense in $L^1(G,M)$ one can find a function $g \in C_c(G,M)$ such that $\|f-g\|_1 < \dfrac{\epsilon}{3\sqrt{2}}$. Let $\xi,\chi \in \widehat{G}$ be such that $\xi^{-1}\chi \in N_k\left(\dfrac{\epsilon}{6\sqrt{2}(1+\|g\|_1)}\right)$. We have

$$\begin{aligned}
\|\hat{f}(\xi)-\hat{f}(\chi)\|_\phi &\leqslant \sqrt{2}\left(\|\hat{f}(\xi)-\hat{g}(\xi)\|_\phi+\|\hat{g}(\xi)-\hat{g}(\chi)\|_\phi+\|\hat{f}(\chi)-\hat{g}(\chi)\|_\phi\right)\\
&\leqslant \sqrt{2}\left(2\|f-g\|_1+\|\hat{g}(\xi)-\hat{g}(\chi)\|_\phi\right)\\
&\leqslant 2\sqrt{2}\|f-g\|_1+\frac{\epsilon}{3}<\epsilon.
\end{aligned}$$

It follows that $\hat{f}$ is continuous on $\widehat{G}$ and there is a sequence $(\hat{g}_n)_n$ which converges to $\hat{f}$ in the norm $\|\cdot\|_1$. The uniform convergence holds since

$$\|\hat{f}-\hat{g}_n\|_\infty \leqslant \sup_{\xi\in\widehat{G}}\|\hat{f}(\xi)-\hat{g}_n(\xi)\|_\phi \leqslant \|\hat{f}-\hat{g}_n\|_1 \to 0 \text{ as } n\to+\infty.$$

Thus, there exists $n_0\in\mathbb{N}$ such that $\|\hat{f}-\hat{g}_n\|_\infty < \dfrac{\epsilon}{\sqrt{2}}$ on every compact subset of $\widehat{G}$, in particular on $\hat{K}_{n_0} = supp(\hat{g}_{n_0})$, the compact support of $\hat{g}_{n_0}$. Let $\xi \in \widehat{G}\setminus\hat{K}_{n_0}$, we have

$$\begin{aligned}
\|\hat{f}(\xi)\|_\phi &\leqslant \sqrt{2}\left(\|\hat{f}(\xi)-\hat{g}_{n_0}(\xi)\|_\phi+\|\hat{g}_{n_0}(\xi)\|_\phi\right)=\sqrt{2}\|\hat{f}(\xi)-\hat{g}_{n_0}(\xi)\|_\phi\\
&\leqslant \sqrt{2}\|\hat{f}-\hat{g}_{n_0}\|_\infty<\epsilon.
\end{aligned}$$

Therefore, $\hat{f}\in C_0(\widehat{G},M)$. □

Proposition 3.3. *Let G be a compact abelian group and M be a quasi-inner product A-module. If $f,g\in L^2(G,M)$, then*

$$\Psi(\hat{f},\hat{g})=\Phi(f,g). \tag{3.1}$$

Proof. Let $f,g\in L^2(G,M)$. We have

$$\begin{aligned}
\Psi(\hat{f},\hat{g}) &= \int_{\widehat{G}}\phi(\hat{f}(\xi),\hat{g}(\xi))\,d\Lambda(\xi)\\
&= \int_{\widehat{G}}\phi\left(\int_G f(u)\overline{\xi(u)}\,d\lambda(u),\int_G g(v)\overline{\xi(v)}\,d\lambda(v)\right)d\Lambda(\xi)\\
&= \int_{\widehat{G}}\int_G\int_G \phi(f(u)\overline{\xi(u)},g(v)\overline{\xi(v)})\,d\lambda(u)\,d\lambda(v)\,d\Lambda(\xi)\\
&= \int_{\widehat{G}}\int_G\int_G \phi(f(u),g(v))\xi(u)\overline{\xi(v)}\,d\lambda(u)\,d\lambda(v)\,d\Lambda(\xi)\\
&= \int_G\int_G \phi(f(u),g(v))\left(\int_{\widehat{G}}\xi(u)\overline{\xi(v)}\,d\Lambda(\xi)\right)d\lambda(u)\,d\lambda(v)\\
&= \int_G\int_G \phi(f(u),g(v))\delta_{u,v}\,d\lambda(u)\,d\lambda(v)\\
&= \int_G \phi(f(u),g(u))\,d\lambda(u)\\
&= \Phi(f,g).
\end{aligned}$$

□

Proposition 3.4. *Let G be a discret abelian group and let (M,ϕ) be a quasi-inner product A-module. The map $L^2(G,M)\to L^2(\widehat{G},M)$, $f\mapsto \hat{f}$ is a $\mathbb{C}$-linear, A-linear and isometric map.*

Proof. The linearity is immediate. Let $f\in L^1(G,M)\cap L^2(G,M)$. From the fact that $\widehat{G}$ is compact set since G is discret and $\Lambda(\widehat{G})<\infty$, we have

$$\begin{aligned}
\int_{\widehat{G}}\|\hat{f}(\xi)\|_\phi^2 d\Lambda(\xi) &= \int_{\widehat{G}}\|\phi(\hat{f}(\xi),\hat{f}(\xi))\|_A d\Lambda(\xi)\\
&\leqslant \int_{\widehat{G}}\|\phi\left(\int_G f(u)\overline{\xi(u)}d\lambda(u),\int_G f(v)\overline{\xi(v)}d\lambda(v)\right)\|_A d\Lambda(\xi)\\
&= \int_{\widehat{G}}\|\int_G\int_G \phi\big(f(u),f(v)\big)\xi(u)\overline{\xi(v)}d\lambda(u)d\lambda(v)\|_A d\Lambda(\xi)\\
&\leqslant \int_{\widehat{G}}\left(\int_G\int_G\|\phi\big(f(u),f(v)\big)\xi(u)\overline{\xi(v)}\|_A d\lambda(u)d\lambda(v)\right)d\Lambda(\xi)\\
&= \int_{\widehat{G}}\left(\int_G\int_G\|\phi\big(f(u),f(v)\big)\|_A|\xi(u)\overline{\xi(v)}|d\lambda(u)d\lambda(v)\right)d\Lambda(\xi)\\
&= \int_{\widehat{G}}\left(\int_G\int_G\|\phi\big(f(u),f(v)\big)\|_A d\lambda(u)d\lambda(v)\right)d\Lambda(\xi)\\
&= \Lambda(\widehat{G})\int_G\int_G\|\phi\big(f(u),f(v)\big)\|_A d\lambda(u)d\lambda(v)\\
&\leqslant 2\Lambda(\widehat{G})\int_G\int_G\|f(u)\|_\phi\|f(v)\|_\phi d\lambda(u)d\lambda(v)\\
&\leqslant 2\Lambda(\widehat{G})\left(\int_G\|f(u)\|_\phi d\lambda(u)\right)\left(\int_G\|f(v)\|_\phi d\lambda(v)\right)\\
&= 2\Lambda(\widehat{G})\left(\int_G\|f(u)\|_\phi d\lambda(u)\right)^2<\infty.
\end{aligned}$$

Hence, $\hat{f}\in L^2(\widehat{G},M)$. Furthermore, given that $L^2(G,M)\cap L^1(G,M)$ is dense in $L^2(G,M)$, one can extend the Fourier transform to the map $L^2(G,M)\to L^2(\widehat{G},M)$, $f\mapsto\hat{f}$. And for all $f\in L^2(G,M)$ we have

$$\begin{aligned}
\|\hat{f}\|_\Psi^2 &= ||\Psi(\hat{f},\hat{f})\|_A=\|\Phi(f,f)\|_A\\
&= \|f\|_\Phi^2.
\end{aligned}$$

□

Proposition 3.5. *Let G be a loally compact abelian group and (M,ϕ) be a quasi-inner product A-module. If $f\in L^2(G,M)$ and $g\in L^2(\widehat{G},M)$, then*

$$\Phi(f,\breve{g})=\Psi(\hat{f},g). \tag{3.2}$$

Proof. Let $f \in L^2(G, M)$ and $g \in L^2(\widehat{G}, M)$, we have

$$\begin{aligned}
\Phi(f, \check{g}) &= \int_G \phi(f(u), \check{g}(u)) d\lambda(u) \\
&= \int_G \phi\left(f(u), \int_{\widehat{G}} g(\xi)\xi(u) d\Lambda(\xi)\right) d\lambda(u) \\
&= \int_G \int_{\widehat{G}} \phi(f(u), g(\xi)\xi(u)) d\Lambda(\xi) d\lambda(u) \\
&= \int_G \int_{\widehat{G}} \phi(f(u)\overline{\xi(u)}, g(\xi)) d\Lambda(\xi) d\lambda(u) \\
&= \int_{\widehat{G}} \phi\left(\int_G f(u)\overline{\xi(u)} d\lambda(u), g(\xi)\right) d\Lambda(\xi) \\
&= \int_{\widehat{G}} \phi(\hat{f}(\xi), g(\xi)) d\Lambda(\xi) \\
&= \Psi(\hat{f}, g).
\end{aligned}$$

□

4. Conclusion

In this paper, we have considered a locally compact abelian group G and a quasi-inner product C^*-module M. We have looked into a few important Fourier transform properties for the Bochner integrable M-valued functions on G. After, we will examin the compact groups framework.

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