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Szeged and vertex PI Index of Graphs over Modules

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Abstract. If we have a commutative ring R and it is identified as $1 \neq 0$ there is an R-module M, $G_R(M)$ will denote the Scalar product graph of M. Vertices of $G_R(M)$ are elements of M, and a, b in M are adjoining if $a = rb$ or $b = ra$ for some $r \in R$. In this paper, we investigate topological indices of the Scalar product graphs on some modules.

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1. Introduction

Throughout the paper, we only consider simple connected graphs. Let G be a graph with sets $V(G)$ and $E(G)$ as vertex and edge set. For two arbitrary vertices, a and v of G, the distance between a and b, denoted by $d(u, v)$, is the length of the shortest path between them. For a graph G, the degree of a vertex v, denoted by $deg(v)$, is the number of edges incident to v. Any function on a graph that doesn't have a relation to

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the labeling of its vertices is a graph invariant. Those graph invariants based on distances are applicable in chemistry. The topological index is a graph invariant. The Wiener index is the first topological index defined by chemist Wiener [17].

Another topological index is the Szeged index, denoted by $Sz(G)$, introduced by Ivan Gutman[6]. Let $e = uv$ be an edge and $n_u(e \mid G)$, $n_v(e \mid G)$ are the size of the set of vertices which be closer to u and to v, respectively. Vertices with equal distance from u and v are not considered. The Szeged index of the G is defined as:

$$Sz(G) = \sum_{e=uv \in E(G)} n_u(e \mid G) n_v(e \mid G) \tag{1.1}$$

One of important indices in graph is $PI_v(G)$. For every $e \in E(G)$, Let the sum of $[m_{eu}(e \mid G) + m_{ev}(e \mid G)]$, where $m_{eu}(e \mid G)$, $m_{ev}(e \mid G)$ are the size of the set of vertices which be closer to u than v and, v than u, respectively. This index is named vertex PI index of G. We suppose, there are two separate graphs as G and H. One of the operation between the two graphs is $G+H$. This is a graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ as vertices and edge sets. This operation is named the join of G and H.

Afkhami et al.[1] introduced one the most essential graphs over the ring, the cozero-divisor graph of ring R, denoted by $\Gamma'(R)$. Let $W(R)$ is the set of all non-unit elements of R and $W(R)^* = W(R) \setminus \{0\}$. The vertices of $\Gamma'(R)$ are $W(R)^*$ and edges of this graph are $\{e = ab : \ a \notin Rbandb \notin Ra\}$.

Nouri-Jouybari et al.[11], defined the Scalar product graph of modules, denoted by $G_R(M)$. Vertices of this graph is M, and a, b in M are adjoining if $a = rb$ or $b = ra$ for some $r \in R$. They state some properties of these graphs. For more information about graphs see: [4], [15], [16]. In [12], [13] authors computed Wiener and first Zagreb indices of the scalar product graph of $\mathbb{Z}_{2p}, \mathbb{Z}_{3p}, \mathbb{Z}_{5p}$.

In the next section, we express some properties of scalar product graphs of some $\mathbb{Z}$-modules by the join of two graphs. In section 3, we see another definition of Szeged and PI indices of graphs and present some formulas for computing Szeged and PI indices of the scalar product graphs of some modules.

2. Scalar product graph of modules

In this section, first, we see some primary properties of the scalar product graphs on some modules, then express computing topological indices of these graphs.

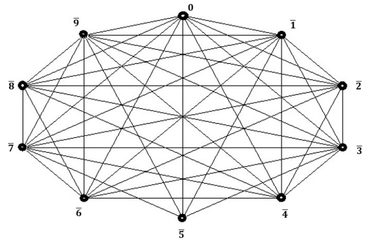

Figure 1. Scalar-product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{10}$

Definition 2.1 ([11])**.** For a commutative ring R with identity $1 \neq 0$ and an R-module M, let $G_R(M)$ denote the Scalar product graph of M. $G_R(M)$ is a graph with vertices in M and a, b in M are adjoining if $a = rb$ or $b = ra$ for some $r \in R$. See Fig1.

Remark 2.2. If M be an R-module, then for $G_R(M)$, We have, $a, b \in M$ are two adjacent vertices if and only if $Ra \subseteq Rb$ or $Rb \subseteq Ra$.

Remark 2.3. As for definition of a cozero-divisor graph over modules, we have:
(1) Vertices $W_R(M)^*$ of $G_R(M)$ make the complement of a cozero-divisors graph of M.
(2) We have $G_R(M) = G_1 + G_2$, That is, G_1 is a complete graph and G_2 is the complement of a cozero-divisor graph of M.

Definition 2.4. [17] The Wiener index of a graph G is defined as:

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d(u,v) \tag{2.1}$$

respectively.

Theorem 2.5. [12] *Suppose $p \geq 3$ is a prime number. Then we have $W(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) = 2p^2 - 1$.*

Theorem 2.6. [12] *Suppose $p \geq 5$ is a prime number. Then we have $W(G_{\mathbb{Z}}(\mathbb{Z}_{3p})) = \frac{9}{2}p^2 + \frac{1}{2}p - 2$.*

Theorem 2.7. [12] *Suppose p is a prime number. Then we have $W(G_{\mathbb{Z}}(\mathbb{Z}_{5p})) = \frac{25}{2}p^2 + \frac{3}{2}p - 4$.*

Definition 2.8. [5] The first and second Zagreb indices of a graph G are defined as:

$$M_1(G) = \sum_{v \in V(G)} deg(v)^2, \tag{2.2}$$

$$M_2(G) = \sum_{uv \in E(G)} deg(v)deg(v) \tag{2.3}$$

respectively.

Theorem 2.9. [13] *Suppose that $p \geq 3$ is a prime number, and G is the Scalar product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{2p}$. Then $M_1(G) = 8p^3 - 15p^2 + 13p - 4$.*

Theorem 2.10. [13] *Suppose that $p \geq 5$ is a prime number, and G is the Scalar product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{3p}$. Then $M_1(G) = 27p^3 - 40p^2 + 25p - 4$.*

Theorem 2.11. [14] *Suppose that $p \geq 3$ is a prime number, and G is the Scalar product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{2p}$. Then $M_2(G) = 8p^4 - 22p^3 + \frac{55}{2}p^2 - \frac{33}{2}p + 4$.*

Definition 2.12. [7] The Harary index of a graph G is defined as:

$$H(G) = \sum_{v_i, v_j \in V(G), i \neq j} \frac{1}{d(v_i, v_j)}, \tag{2.4}$$

Theorem 2.13. [14] *Suppose that $p \geq 3$ is a prime number, and G is the Scalar product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{2p}$. Then $H(G) = 2p^2 - \frac{3}{2}p + \frac{1}{2}$.*

Theorem 2.14. [14] *Suppose that $p \geq 5$ is a prime number, and G is the Scalar product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{3p}$. Then $H(G) = \frac{9}{2}p^2 - \frac{5}{2}p + 1$.*

Theorem 2.15. [14] *Suppose that $p \geq 7$ is a prime number, and G is the Scalar product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{5p}$. Then $H(G) = \frac{25}{2}p^2 - \frac{9}{2}p + 2$.*

3. Szeged and vertex PI indices of graphs

In this section, first, we see the definition of Szeged and vertex PI indices of graphs, then computing these indices for the Scalar product of some $\mathbb{Z}$-modules by joining two graphs.

Let $e = uv \in E(G)$ and define the partition $\{N_u(e), N_v(e), N_0(e)\}$ of the vertices of G with respect to e as follows:

$$N_u = \{w \in V : d(u,w) < d(v,w)\} \tag{3.1}$$

$$N_v = \{w \in V : d(v,w) < d(u,w)\} \tag{3.2}$$

$$N_0 = \{w \in V : d(u,w) = d(v,w)\} \tag{3.3}$$

We denote $n_u(e), n_v(e)$ and $n_0(e)$ for size of $N_u(e), N_v(e)$ and $N_0(e)$, respectively.

Definition 3.1. [10] There is a definition of Szeged and vertex PI index which is as follows:

$$Sz(G) = \sum_{e=uv\in E(G)} n_u(e \mid G)n_v(e \mid G) \tag{3.4}$$

$$PI_v(G) = \sum_{e=uv\in E(G)} n_u(e \mid G) + n_v(e \mid G) \tag{3.5}$$

Theorem 3.2. [10] *Let $G_1, \dots, G_n$ be graphs. Then, we have:*

$$Sz(G_1 + \dots + G_n) = \frac{1}{2}\sum_{i=1}^{n} Sz(G_i + G_i) + \frac{1}{2}(\sum_{i=1}^{n} \mid V_i \mid^2 -2 \mid E_i \mid)^2 \tag{3.6}$$

$$-\sum_{i=1}^{n}(\mid V_i \mid^2 -2 \mid E_i \mid)^2 \tag{3.7}$$

Corollary 3.3. *Let G_1, G_2 be two graphs with $V_i = |V(G_i)|$, $E_i = |E(G_i)|$, $i = 1, 2$. Then:*

$$\begin{aligned} Sz(G_1 + G_2) =& \frac{1}{2}[Sz(G_1 + G_1) + Sz(G_2 + G_2)] \\ &+ \frac{1}{2}[\mid V_1 \mid^2 -2 \mid E_1 \mid) + (\mid V_2 \mid^2 -2 \mid E_2 \mid)]^2 \\ &- \frac{1}{2}[(\mid V_1 \mid^2 -2 \mid E_1 \mid)^2 + (\mid V_2 \mid^2 -2 \mid E_2 \mid)^2] \end{aligned}$$

Theorem 3.4. *Suppose that $p \geq 3$ is a prime number, and G is the Scalar product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{2p}$. Then $Sz(G) = 4p^2 - 4p + 1$.*

Proof. By remark 2.3, we have $G_{\mathbb{Z}}(\mathbb{Z}_{2p}) = K_p + \overline{K_{1,p-1}}$. Due to the above corollary 3.3, we have:

$$\begin{aligned} Sz(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) &= Sz(K_p + \overline{K_{1,p-1}}) \\ &= \frac{1}{2}[Sz(K_p + K_p) + Sz(\overline{K_{1,p-1}} + \overline{K_{1,p-1}})] \\ &+ \frac{1}{2}[\mid V(K_p) \mid^2 -2 \mid E(K_p) \mid) + (\mid V(\overline{K_{1,p-1}}) \mid^2 \\ &- 2 \mid E(\overline{K_{1,p-1}}) \mid)]^2 - \frac{1}{2}[(\mid V(K_p) \mid^2 -2 \mid E(K_p) \mid)^2 \\ &+ (\mid V(\overline{K_{1,p-1}}) \mid^2 -2 \mid E(\overline{K_{1,p-1}}) \mid)^2] \end{aligned}$$

Therefore, we have:
$Sz(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) = 4p^2 - 4p + 1.$ □

Theorem 3.5. [10] *Let $G_1, \ldots, G_n$ be graphs. Then, we have:*

$$PI_v(G_1 + \ldots + G_n) = (\sum_{i=1}^{n} | V_i |)(\sum_{i=1}^{n}(| V_i |^2 - 2 | E_i |)) - 2(\sum_{i=1}^{n} | V_i | (| V_i |^2 - 2 | E_i |)) + \frac{1}{2}\sum_{i=1}^{n} PI_v(2G_i)$$

Corollary 3.6. *Let G_1, G_2 be two graphs with $V_i = |V(G_i)|$, $E_i = |E(G_i)|$, $i = 1, 2$. Then:*

$$\begin{aligned} PI_v(G_1 + G_2) =& (| V_1 | + | V_2 |)[(| V_1 |^2 - 2 | E_1 |) + (| V_2 |^2 - 2 | E_2 |)] \\ & - 2[| V_1 | (| V_1 |^2 - 2 | E_1 |) + | V_2 | (| V_2 |^2 - 2 | E_2 |)] \\ & + \frac{1}{2}[PI_v(2G_1) + PI_v(2G_2)] \end{aligned}$$

Theorem 3.7. *Suppose that $p \geq 3$ is a prime number, and G is the scalar product graph of $\mathbb{Z}$-modules $\mathbb{Z}_{2p}$. Then $Sz(G) = 6p^2 - 6p + 2$.*

Proof. By remark 2.3, we have $G_{\mathbb{Z}}(\mathbb{Z}_{2p}) = K_p + \overline{K_{1,p-1}}$. Due to the above corollary 3.6, we have:

$$\begin{aligned} PI_v(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) &= Sz(K_p + \overline{K_{1,p-1}}) \\ &= (|V(K_p)| + \left|V(\overline{K_{1,p-1}})\right|)[(|V(K_p)|^2 - 2\,|E(K_p)|) \\ &+ (\left|V(\overline{K_{1,p-1}})\right|^2 - 2\left|E(\overline{K_{1,p-1}})\right|)] - 2[|V(K_p)|\,(|V(K_p)|^2 \\ &- 2\,|E(K_p)|) + \left|V(\overline{K_{1,p-1}})\right| (\left|V(\overline{K_{1,p-1}})\right|^2 - 2\left|E(\overline{K_{1,p-1}})\right|)] \\ &+ \frac{1}{2}[PI_v(2K_p) + PI_v(2\overline{K_{1,p-1}})] \end{aligned}$$

On the other hand, we have $PI_v(K_n) = n(n-1)$, therefore, we have: $PI_v(G_{\mathbb{Z}}(\mathbb{Z}_{2p})) = 6p^2 - 6p + 2$. □

4. Conclusion

According to the new definition of a type of graph on modules, other states of this graph can be considered for the rest of the modules other than $\mathbb{Z}$-modules and topological indices can be measured on it. New formulas are available for other topological indices.

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