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Stability and boundedness for a certain second order integro-differential equation

Melek Gözen[1]
[1] Department of Business Administration, Faculty of Management, Van Yuzuncu Yil University, 65080, Erciş–Van, TURKEY

Abstract. In theory applications of second order integro-differential equations, it is crucial to investigate qualitative features of solutions such as stability, boundedness, and etc. There is an extensive literature regarding these qualitative behaviors of solutions of second order ordinary differential equations. These qualitative properties of solutions of second order ordinary differential equations have been extensively studied in the literature. Despite this case, the literature on these qualitative aspects of second-order integro-differential equations is somewhat limited. In this paper, we obtained some new criteria for the stability and boundedness of solutions to a certain second order nonlinear integro-differential equation (IDE). By defining and then using a suitable Lyapunov function (LF), we are able to establish the asymptotic stability and boundedness of solutions of that IDE. In particular cases of the IDE, two examples on the stability and boundedness of solutions are given. The results of this study extend and improve some recent results of the literature and have new contributions to the qualitative theory of IDEs.

Keywords: Stability, boundedness, second order, integro-differential equations.

[1]Corresponding author: melekgozen2013@gmail.com
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1. Introduction

As we know from the pertinent literature, ordinary differential equations (ODEs) of second order have a wide range of real-world applications. For some applications, see, for examples the books of Ahmad and Rama Mohana Rao [4], Bellman [6], Burton [9], Reissig et al. [16], Yoshizawa [24]. Indeed, there is an extensive literature on the qualitative behaviors called stability, boundedness, convergence, etc. of solutions of ordinary differential equations of second order, see, for instance, the papers of Adams et al. [1], Adeyanju ([2],[3]), Athanassov [5], Bihari [8], Burton and Grimmer [10], Chang [11], Gözen [12], Graef and Spikes [13], Hatvani[14], Lalli [15], Sugie and Amano [17], Tunç ([18] [19]), Tunç and Tunç ([20],[21]), Wong [22], Yang [23], and Zarghamee and Mehri [25].

We will now summarize briefly a few works on the stability, boundedness, etc. of certain ODEs of second order.

In 1969, Lalli [15] discussed stability and boundedness of solutions of the following ODE of second order for the homogeneous and nonhomogenous cases, respectively:

$$\big(r(t)u'\big)' + a(t)f(u)g(u') = q(t).$$

Later, Zarghamee and Mehri [25] gave some results on the stability and boundedness of solutions to a class of second order differential equation of the form:

$$\frac{d}{dt}\big(r(t)u'\big) + a(t)f(u)g(u') + b(t)h(u)m(u') = 0.$$

In a recent paper, Adams et al. [1] derived some sufficient criteria for the stability and boundedness of solutions to the below second order nonlinear differential equation when $p(t,x,x') = 0$ and $p(t,x,x') \neq 0$, respectively:

$$x'' + b(t)f(x,x') + c(t)g(x)h(x') = p(t,x,x').$$

In Adams et al. [1] , a suitable Lyapunov function is used as a basic tool to prove the results therein.

In this paper, inspired by the aforementioned works, especially by the results of Adams et al. [1] and that can be seen in the references of this paper, and in the database of the relevant literature, we will dealt with second order nonlinear IDE as follows:

$$\begin{aligned} & x'' + a(t)f(t,x,x') + b(t)g(x)h(x') + c(t)g_1(x') + d(t)g_2(x) \\ & = \int\limits_0^t K(t,s)x'(s)ds + p(t,x,x'), \end{aligned} \tag{1.1}$$

where $a, b, c, d \in C\left(R^+, (0,\infty)\right)$, $R^+ = [0,\infty)$, $f \in C\left(R^+ \times R^2, R\right)$, $f(t,x,0) = 0$, g, g_1, g_2, $h \in C\left(R, R\right)$, $g(0) = g_1(0) = g_2(0) = 0$, $p \in C\left(R^+ \times R^2, R\right)$, and $K \in C\left(R^+ \times R^+, R\right)$. The aim of this study is to investigate the stability and boundedness of solutions to the IDE (1.1) for the cases $p(t,x,x') = 0$ and $p(t,x,x') \neq 0$, respectively. The basic technique in the proofs will be a suitable new Lyapunov function to prove the new results of this study. Hence, we will improve and extend the recent results of Adams et al. [1] and do new contributions to some results of the above sources.

2. Preliminaries

We will have an occasion to use the following well-known lemma due to Bellman (see [Ahmad and Rama Mohana Rao [4],see also Bellman [6]), which is also known as Gronwall's inequality.

Lemma 2.1 (Gronwall-Reid Bellman inequality). *Let c be a nonnegative constant and let u and v be nonnegative continuous functions on some interval $t_0 \leq t \leq t_0 + a$ satisfying*

$$u(t) \leq c + \int_{t_0}^t u(s)v(s)ds, t \in [t_0, t_0 + a].$$

Then, the inequality

$$u(t) \leq cexp[\int_{t_0}^t v(s)ds], t \in [t_0 + t_0 + a],$$

holds.

We will now consider the differential system as follows:

$$X' = F(t,X), \tag{2.1}$$

where $t \in \mathbb{R}^+$, $X \in R^n$, $F \in C\left(R^+ \times R^n, R^n\right)$ and $F(t,0) = 0$.

Theorem 2.2 (Yoshizawa [24]). *Assume that there exists a function $V(t,X)$ defined for $t \geq 0$, $|X| < \delta$(δ is a positive constant) continuous with the following properties:*

(i) $V(t,0) \equiv 0$;

(ii) $V(t,X) \geq a(|X|)$, *where $a(r)$ is continuous and monotonically increasing, $a(0) = 0$;*

(iii) $V'(t,X) \leq -c(|X|)$, *where* $c(r)$ *is continuous on* $[0,\delta]$ *and positive, and if* $F(t,X)$ *is bounded, then zero solution of Eq.* (2.1) *is asymptotically stable.*

3. Main results

Let $x' = y$. Subsequently, we transform the IDE (1.1) to the equivalent system as follows:

$$\begin{aligned} &x' = y,\\ &y' = -a(t)f(t,x,y) - b(t)g(x)h(y) - c(t)g_1(y)\\ &- d(t)g_2(x) + \int_0^t K(t,s)y(s)ds. \end{aligned} \tag{3.1}$$

Basic Assumptions

The following are the basic assumptions to formulate the qualitative results for the IDE (1.1)

(C1) Let $a_0,\ b_0,\ c_0$ and d_0 be positive constants such that $a(t) \geq a_0 \geq 0, b(t) \geq b_0, c(t) \geq c_0, d(t) \geq d_0,\ d'(t) \leq 0$ and $c'(t) \leq 0,\ t \in R^+$, $g(x) > 0$, $(x \neq 0)$, $x \in R$, $g(0) = 0$, where $a,\ b,\ d \in C(R^+, R^+)$, $g \in C(R,R)$, $c \in C^1(R^+, R^+)$;

(C2)

$$f \in C(R^+ \times R^2, R),\ g_2 \in C(R,R),\ h,\ g_1 \in C(R,R);$$

(C3)

$$\frac{f(t,x,y)}{y} \geq \eta,\ (y \neq 0),\ \eta > 0,\ \eta \in R,$$

$$f(t,x,0) = 0,\ (t,x,y) \in R^+ \times R^2, g_1(0) = g_2(0) = 0$$

(C4)

$$\frac{g_1(y)}{y} \geq \alpha,\ (y \neq 0), \frac{g_2(x)}{x} \geq \gamma,\ (x \neq 0),$$

$$\alpha,\ \gamma > 0,\ \alpha,\ \gamma \in R, x, y \in R;$$

(C5)

$$\frac{h(y)}{y} \geq \beta,\ (y \neq 0), h(0) = 0, \beta > 0, \beta \in R, y \in R;$$

(C6)

$$\int_0^t |K(u,t)|\, du \leq K_0, \int_0^t |K(t,s)|\, ds \leq K_1, K_0,\ K_1 > 0,\ t \in R^+.$$

(C7)
$$-\frac{1}{2}K_0 + a(t)n + c(t)\alpha - \frac{1}{2}K_1 + b(t)\beta g(x) \geq \delta, \delta > 0.$$

First, we establish the stability result of this study when $p(t,x,x') = 0$.

Theorem 3.1. *In addition to the conditions (C1)-(C6), we assume that there exists a positive constant ξ such that $\lim_{t\to\infty} c(t) = \xi$. Then, the zero solution of the system of IDEs* (3.1) *is asymptotically stable in the sense of Lyapunov.*

Proof. We define the Lyapunov function:
$$V(t,x,y) = d(t)\int_0^x g_2(s)ds + \frac{1}{2}y^2 + \frac{1}{2}\int_0^t \int_t^\infty |K(u,s)|\, y^2(s)duds \quad (3.2)$$
In view of the conditions (C1), (C4) and (C6) we get
$$\begin{aligned} V(t,x,y) &\geq \frac{1}{2}d_0\gamma x^2 + \frac{1}{2}y^2 \\ &\geq A_1(x^2+y^2), \end{aligned}$$
where $A_1 = \min\left\{\frac{1}{2}d_0\gamma, \frac{1}{2}\right\}$.

Let $V' = \frac{d}{dt}V(t,x,y)$. By differentiating the Lyapunov function (3.2) with respect to t along the system of IDEs (3.1) we have :
$$\begin{aligned} V' =& d'(t)\int_0^x g_2(s)ds + d(t)x'g_2(x) + \frac{1}{2}\int_t^\infty |K(u,t)|\, y^2(t)du \\ &- \frac{1}{2}\int_0^t |K(t,s)|\, y^2(s)ds + yy' \\ =& d'(t)\int_0^x g_2(s)ds + d(t)yg_2(x) + \frac{1}{2}\int_t^\infty |K(u,t)|\, y^2(t)du \\ &- \frac{1}{2}\int_0^t |K(t,s)|\, y^2(s)ds - a(t)yf(t,x,y) \\ &- b(t)g(x)yh(y) - c(t)yg_1(y) \\ &- d(t)yg_2(x) + y\int_0^t K(t,s)y(s)ds. \end{aligned}$$

Using the conditions (C1)-(C6) and the inequality $0 \le (x-y)^2$, we have

$$\begin{aligned}
V' \le & \frac{1}{2}d'(t)\gamma x^2 + \frac{1}{2}y^2(t)\int_0^\infty |K(u,t)|\,du - \frac{1}{2}\int_0^t \left|K(t,s)y^2(s)ds\right| \\
& - a(t)\eta y^2 - b(t)g(x)\beta y^2 - c(t)\alpha y^2 + y\int_0^t K(t,s)y(s)ds \\
\le & \frac{1}{2}d'(t)\gamma x^2 - b(t)g(x)\beta y^2 + \frac{1}{2}y^2K_0 - a(t)\eta y^2 - c(t)\alpha y^2 + \frac{1}{2}y^2K_1 \\
\le & \frac{1}{2}d'(t)\gamma x^2 + [\frac{1}{2}K_0 - a(t)\eta - c(t)\alpha + \frac{1}{2}K_1 - b(t)\beta g(x)]y^2 \\
\le & \frac{1}{2}d'(t)\gamma x^2 - \delta y^2 \\
\le & 0.
\end{aligned}$$

Thus, all the conditions of Theorem 2.2 hold. Thus, the zero solution of the system of IDEs (3.1) is asymptotically stable. Thereby, this completes the proof of Theorem 3.1.

Second, we will give the boundedness result of this study when $p(t,x,x') \neq 0$. □

Theorem 3.2. *Let us assume that the conditions of Theorem 3.1 hold except* $f(t,x,0)=0$, $h(0)=0$ *and* $g_1(0)=g_2(0)=0$. *In additions, we also assume that*

$$|p(t,x,y)| \le e(t) \ge 0, \int_0^\infty e(s)ds < \infty,$$

where $e(t) \in L^1(0,\infty)$ *and* $L^1(0,\infty)$ *is a space of integrable Lebesque functions. Then , there exists a positive constant* A_3 *such that all solutions* $(x(t),y(t))$ *the system of IDEs* (3.1) *satisfy*

$$|x(t)| \le A_3, \;\; |y(t)| \le A_3$$

for all $t \in R^+$, $R^+ = [0,\infty)$.

Proof. To prove Theorem 3.2, we use Lyapunov function $V(t,x,y)$ which is given by the Eq. (3.2). For the case $p(t,x,y) \neq 0$, and applying the assumption of Theorem 3.1, we can revise the result of Theorem 3.1 as follows:

$$\begin{aligned}\frac{d}{dt}V(t,x,y) \leq & \frac{1}{2}d'(t)\gamma x^2 - \delta y^2 + yp(t,x,x') \\ \leq & \frac{1}{2}d'(t)\gamma x^2 - \delta y^2 + ye(t) \\ \leq & \frac{1}{2}d'(t)\gamma x^2 - \delta y^2 + |y|\, e(t).\end{aligned}$$

Using the in quality $|y| < 1 + y^2$ and the condition $|p(t,x,y)| \leq e(t)$, we get

$$\begin{aligned}\frac{d}{dt}V(t,x,y) \leq & \frac{1}{2}d'(t)\gamma x^2 - \delta y^2 + (1+y^2)e(t) \\ \leq & \frac{1}{2}d'(t)\gamma x^2 - \delta y^2 + (1 + A_1^{-1}V(t,x,y))e(t) \\ \leq & (1 + A_1^{-1}V(t,x,y))e(t) \\ = & e(t) + A_1^{-1}V(t,x,y)e(t), \end{aligned} \tag{3.3}$$

where $y^2 \leq A_1^{-1}V(t,x,y)$. Integrating the inequality (3.3) from 0 to t and using the Gronwall-Bellman lemma, we have

$$\int_0^t V'(s,x,y)ds \leq \int_0^t e(s)ds + \int_0^t A_1^{-1}V(s,x,y)e(s)ds$$

so that

$$\begin{aligned}V(t,x,y) \leq & V(0,x(0),y(0)) + \int_0^t e(s)ds + \int_0^t A_1^{-1}V(s,x,y)e(s)ds \\ \leq & V(0,x(0),y(0)) + \int_0^\infty e(s)ds + \int_0^\infty A_1^{-1}V(s,x,y)e(s)ds \\ \leq & [V(0,x(0),y(0)) + B]\exp\left(A_1{}^{-1}B\right), \end{aligned} \tag{3.4}$$

where $B = \int\limits_0^\infty e(s)ds$.

In view of the inequalities

$$V(t,x,y) \geq \frac{1}{2}d_0\gamma x^2 + \frac{1}{2}y^2$$

and

$$V(t,x,y) \leq [V(0,x(0),y(0)) + B]\exp\left(A_1{}^{-1}B\right),$$

we can conclude the boundedness of all solutions of the system of IDEs (3.1). Thereby, the proof of Theorem 3.2 is now complete.

We will now present two examples in particular cases of the IDE (1.1) that satisfy the conditions of Theorem 3.1 and Theorem 3.2 , respectively. □

Example 3.3. . Consider the second order IDE as follows:

$$x'' + (1+t^2)x'e^{x^2+t^2} + (1+e^{-t^2})(1+|x|)(1+e^{-x'})x'$$

$$+ (1+e^{-2t})(1+e^{x'})x' + (1+e^{-t})x = \int_0^t e^{-(t+s)}x'(s)ds. \tag{3.5}$$

The equivalent system of IDE (3.5) can be written as follows:

$$\begin{aligned} x' =& y, \\ y' =& - \ (1+t^2)xye^{x^2+t^2} - (1+e^{-t^2})(1+|x|)(1+e^{-y})y \\ & - \ (1+e^{-2t})(1+e^{y})y - (1+e^{-t})x + \int_0^t e^{-(t+s)}y(s)ds. \end{aligned} \tag{3.6}$$

It is obvious that the system (3.6) satisfies the conditions (C1) and (C2). We now consider the Lyapunov function given as follows, which is a particular case of that one given by (3.3):

$$V(t,x,y) = \frac{1}{2}(1+e^{-t})x^2 + \frac{1}{2}y^2 + \frac{1}{2}\int_0^t \int_t^\infty |K(u,s)|\, y^2(s)duds$$

The differentiating the Lyapunov $V \equiv V(t,x,y)$ along the system (3.6), we get

$$V' = -\frac{1}{2}e^{-t}x^2 + [-(1+e^{-t^2})(1+|x|) - (1+t^2) - (1+e^{-2t}) + 1]y^2.$$

Thus, there exist a positive constant δ, which is small enough, such that

$$V' \le -\delta(x^2+y^2),$$

which verifies that the asymptotic stability is now established.

Example 3.4. Consider the second order IDE as follows:

$$\begin{aligned} & x'' + (1+t^2)x'e^{x^2+t^2} + (1+e^{-t^2})(1+|x|)(1+e^{-x'})x' \\ & + (1+e^{-2t})(1+e^{x'})x' + (1+e^{-t})x = \int_0^t e^{-(t+s)}x'(s)ds \\ & + \frac{1}{1+t^2+x^4+x'^4}. \end{aligned} \tag{3.7}$$

The discussions of Example 1 hold for the IDE (3.7) except that for the term $\frac{1}{1+t^2+x^4+x'^4}$. Additionally, it is seen that

$$p(t,x,y) = \frac{1}{1+t^2+x^4+y^4}.$$

Hence, regarding to Theorem 3.2, we have that

$$|p(t,x,y)| = \frac{1}{1+t^2+x^4+y^4} \leq \frac{1}{1+t^2} = e(t).$$

Thereby, it follows from the inequality (3.3) that

$$V'(t,x,y) \leq \frac{1}{1+t^2} + A_1^{-1} V(t,x,y) \frac{1}{1+t^2}.$$

In the subsequent, integrating the above inequality from 0 to ∞ and then applying Gronwall-Bellman lemma, we get

$$V(t,x,y) \leq [V(0,x(0),y(0)) + \frac{\pi}{2}] e^{A_1^{-1}\frac{\pi}{2}},$$

where

$$B = \int_0^\infty \frac{1}{1+s^2} ds = \frac{\pi}{2}.$$

From the last inequality and

$$V(t,x,y) \geq \frac{1}{2} d_0 \gamma x^2 + \frac{1}{2} y^2,$$

we can conclude the boundedness of solutions of IDE (3.5).

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